

Higson coronas of coarse spaces.

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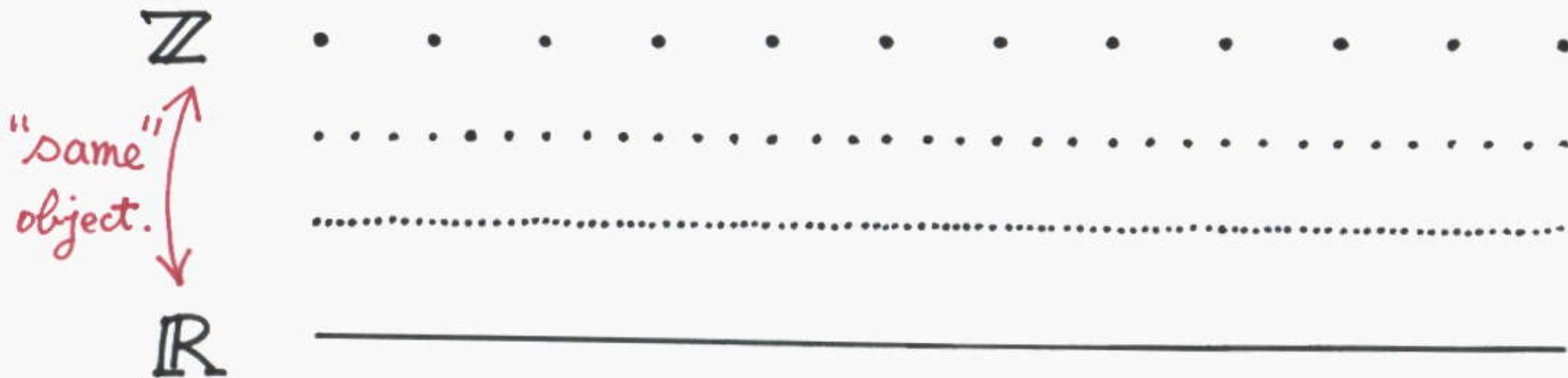
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1. Coarse Geometry and boundaries.

Coarse Geometry (\doteq large-scale/asymptotic geometry)

— looking at metric spaces from a great distance.



1.1 Coarse maps and coarse equivalences.

Definition. X, Y metric spaces.

$f: X \rightarrow Y$ is a coarse map

$\overset{\text{def}}{\iff} \exists \rho: [0, \infty) \rightarrow [0, \infty)$ non-decreasing function s.t.

- $\forall x, x' \in X \quad d(f(x), f(x')) \leq \rho(d(x, x'))$
- $\forall B \subset Y$ bounded $f^{-1}(B)$ is bounded.

$f: X \rightarrow Y$ is a coarse equivalence

$\overset{\text{def}}{\iff}$ • $f: X \rightarrow Y$ is a coarse map

• $\exists g: Y \rightarrow X$ coarse map s.t.

$$g \circ f \doteq id_X, \quad f \circ g \doteq id_Y$$

For $\varphi, \psi: S \rightarrow \mathbb{Z}$, $\varphi \doteq \psi \overset{\text{def}}{\iff} \exists M > 0 \quad \forall s \in S \quad d(\varphi(s), \psi(s)) \leq M$

↑ ↑
set metric space (close)

An equivalent definition of coarse equivalence :

$f: X \rightarrow Y$ coarse equivalence

$\Leftrightarrow \exists \rho_1, \rho_2: [0, \infty) \rightarrow [0, \infty) \quad \exists C > 0 \text{ s.t.}$

• $\rho_1 \leq \rho_2, \lim_{t \rightarrow \infty} \rho_1(t) = \infty$

• $\forall x, x' \in X \quad \rho_1(d(x, x')) \leq d(f(x), f(x')) \leq \rho_2(d(x, x'))$

• $f(X)$ is a C -net in Y (i.e. $\forall y \in Y \quad \exists x \in X \quad d(y, f(x)) \leq C$)

If we can take $\rho_1(t) = A_1 t + B_1, \quad \rho_2(t) = A_2 t + B_2$

for some A_1, A_2, B_1, B_2 , f is called a quasi-isometry.

* f is a quasi-isometry $\xrightleftharpoons{\text{g.i.}}$ f is a coarse eq.

holds for X, Y geodesic spaces.

Examples.

- $f: \mathbb{Z}^n \hookrightarrow \mathbb{R}^n$ is a q.i.

inverse $g: \mathbb{R}^n \rightarrow \mathbb{Z}^n$ is given by $g(x_1, \dots, x_n) = ([x_1], \dots, [x_n])$

- G : a f.g. group $S = S^{-1} \subset G$ finite generating set.

$\Gamma(G, S)$ the Cayley graph of G (w.r.t. S)

vertex set = G ,  $\Leftrightarrow \exists s \in S \quad gs = g'$

$\rightarrow \Gamma(G, S)$ is a geodesic sp. (each edge has length 1.)

\rightarrow induced metric on G is the word metric

$$d_S(g, g') = \min \{ n ; \exists s_1, \dots, s_n \in S \quad gs_1 \cdots s_n = g' \}$$

basic fact: If $S' = S^{-1}$ is another generating set,

$$\Gamma(G, S) \underset{\text{q.i.}}{\sim} (G, d_S) \underset{\text{q.i.}}{\sim} (G, d_{S'}) \underset{\text{q.i.}}{\sim} \Gamma(G, S')$$

1.2 Boundaries

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Coarse geometry $\xrightarrow{\quad}$ Boundaries

on metric spaces $\xleftarrow{?}$ of metric spaces

of metric spaces

?

topological objects.

example.

X : proper

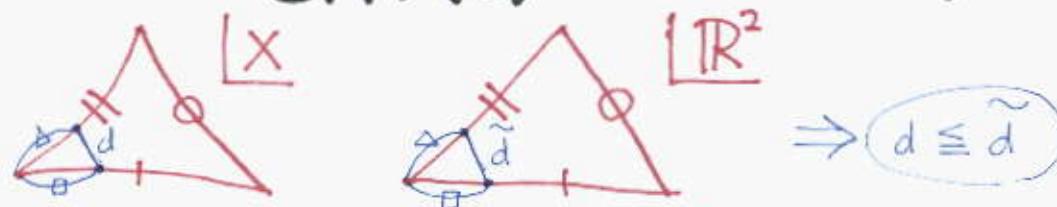


Gromov hyperbolic geodesic

or

CAT(0)

space



$\rightarrow \partial_\infty X$ the geodesic ∂ of X

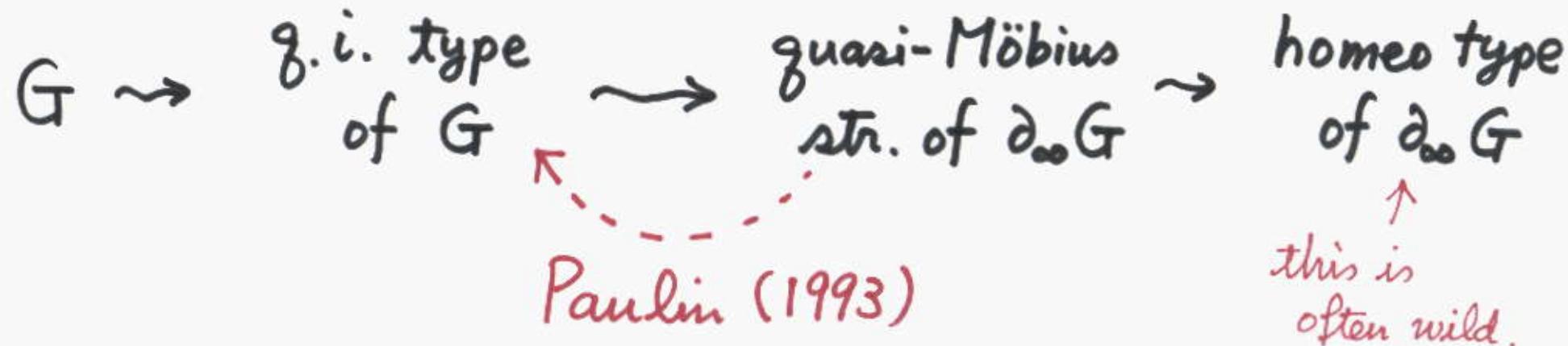
$\gamma \sim \gamma' \Leftrightarrow \exists M > 0 \forall t \geq 0$
 $d(\gamma(t), \gamma'(t)) \leq M$

$\{\gamma: [0, \infty) \rightarrow X \text{ geodesic rays}\} / \sim$

- X, Y proper Gromov hyp. geod. spaces

$f: X \rightarrow Y$ q.i. $\rightarrow \partial_\infty f: \partial_\infty X \xrightarrow{\approx} \partial_\infty Y$ (quasi-Möbius homeo.)

G : Gromov hyp. gp This is not defined for CAT(0).



- G Gromov hyp gp. $\partial_\infty G \approx S^1 (= \partial_\infty \mathbb{H}^2)$

$$\Rightarrow G \underset{\text{q.i.}}{\sim} \mathbb{H}^2$$

(Tukia-Gabai
-Frieden-Casson-Jungreis)

Sometimes we can recover coarse geometry from its ∂ ??



Higson corona is another kind of boundary.

This can be defined in terms of algebra of functions.

X : proper metric space

$$C_b(X) = \{ X \rightarrow \mathbb{R} \text{ bdd conti} \}$$

closed
subalg.

\cup

$$C_p(X) = \left\{ g \in C_b(X); \begin{array}{l} \forall R > 0 \\ \lim_{x \rightarrow \infty} \text{diam } f(B(x, R)) = 0 \end{array} \right\}$$

: the alg of Higson functions

The Higson compactification hX of X is defined by :

$$C(hX) = C_p(X)$$

$\nu X := hX \setminus X$ the Higson corona.

* $X \xrightarrow[f]{\sim} Y$ coarse eq $\Rightarrow \nu X \xrightarrow[\nu f]{\sim} \nu Y$ homeo.

2. Coarse spaces and Higson coronas.

2.1

coarse spaces.

Definition X a set.

$\mathcal{E} \subset \mathcal{P}(X \times X)$ is a coarse structure on X
 \Leftrightarrow (1) $\Delta_X \in \mathcal{E}$

(2) $E \in \mathcal{E} \Rightarrow E^{-1} = \{(y, x); (x, y) \in E\} \in \mathcal{E}$

(3) $E \in \mathcal{E}, F \subset E \Rightarrow F \in \mathcal{E}$

(4) $E, F \in \mathcal{E} \Rightarrow E \cup F \in \mathcal{E}$

(5) $E, F \in \mathcal{E} \Rightarrow E \circ F = \{(x, z); \exists y (x, y) \in E, (y, z) \in F\} \in \mathcal{E}$

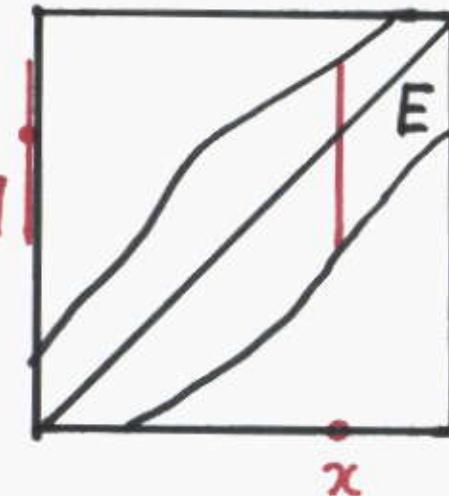
Then $X = (X, \mathcal{E})$: coarse space

each $E \in \mathcal{E}$ is called "controlled set."

$$E[x] = \{y \in X; (x, y) \in E\}$$

: the E -ball about x

$E[x]$



example 1. Bounded coarse structure "standard" coarse str
on a metric space

$X = (X, d)$ proper metric space

$$\hookrightarrow \mathcal{E}_X^b := \left\{ E \subset X \times X ; \begin{array}{l} \exists C > 0 \quad \forall (x, y) \in E \\ d(x, y) \leq C \end{array} \right\}$$

the bounded coarse str. on X

- For $r > 0$, define $E_r := \{(x, y) \in X \times X ; d(x, y) \leq r\}$.

Then $\{E_r ; r > 0\}$ "generates" \mathcal{E}_X^b .

\hookrightarrow " E_r -balls = (closed) r -balls": $E_r[x] = \overline{B}(x, r)$.

- $B \subset X$ is bounded

$$\Leftrightarrow B \times B \in \mathcal{E}_X^b$$

General definition: A subset B of a coarse space $X = (X, \mathcal{E})$ is bounded $\stackrel{\text{def}}{\Leftrightarrow} B \times B \in \mathcal{E}$.

Definition $X = (X, \Sigma)$, $Y = (Y, \mathcal{F})$, coarse spaces
 S : a set.

(1) $\varphi, \psi: S \rightarrow X$ are close ($\varphi \doteq \psi$)

$$\overset{\text{def}}{\iff} \left\{ (\varphi(s), \psi(s)); s \in S \right\} \in \Sigma.$$

(2) $f: X \rightarrow Y$ is a coarse map

$$\overset{\text{def}}{\iff} \left\{ \begin{array}{l} \forall E \in \Sigma \quad (f \times f)(E) = \left\{ (f(x), f(x')); (x, x') \in E \right\} \in \mathcal{F} \\ \text{and} \\ B \subset Y \text{ bounded} \Rightarrow f^{-1}(B) \text{ bounded} \end{array} \right.$$

(3) $f: X \rightarrow Y$ coarse map is a coarse equivalence

$$\overset{\text{def}}{\iff} \exists g: Y \rightarrow X \text{ coarse map } g \circ f \doteq id_X, f \circ g \doteq id_Y$$

These definitions are consistent with the metric definitions
when X, Y metric spaces and $\Sigma = \Sigma_X^b$, $\mathcal{F} = \Sigma_Y^b$.

example 2. C_0 coarse structure

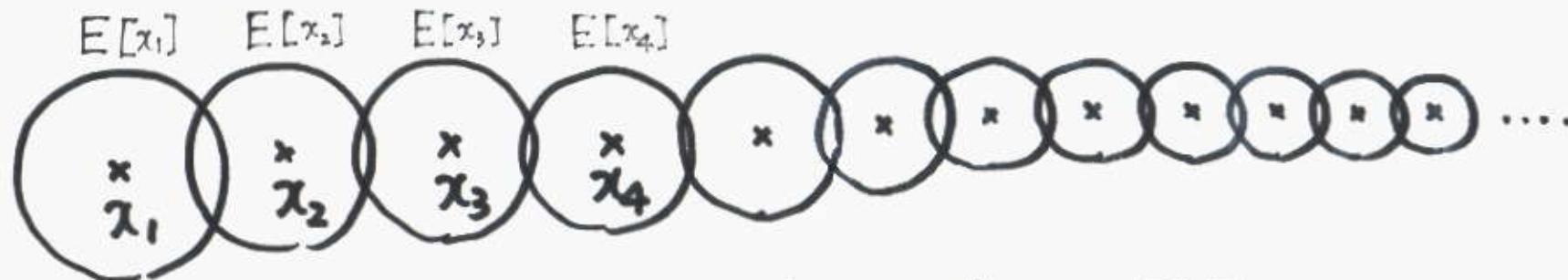
X locally compact metric space

$$\mathcal{E}_X^0 := \left\{ E \subset X^2; \begin{array}{l} \forall \varepsilon > 0 \exists K \subset X \text{ compact} \\ (x, y) \in E \setminus K \times K \Rightarrow d(x, y) < \varepsilon \end{array} \right\}$$

another coarse str
for locally cpt metric spaces

$$E \in \mathcal{E}_X^0$$

: fix



$x_n \rightarrow \infty$ then $\text{diam } E[x_n] \rightarrow 0$

Prop. X, Y locally compact metric, $f: X \xrightarrow{\text{conti.}} Y$

Then $f: (X, \mathcal{E}_X^0) \rightarrow (Y, \mathcal{E}_Y^0) \Leftrightarrow f:$ proper and
coarse map uniformly conti.

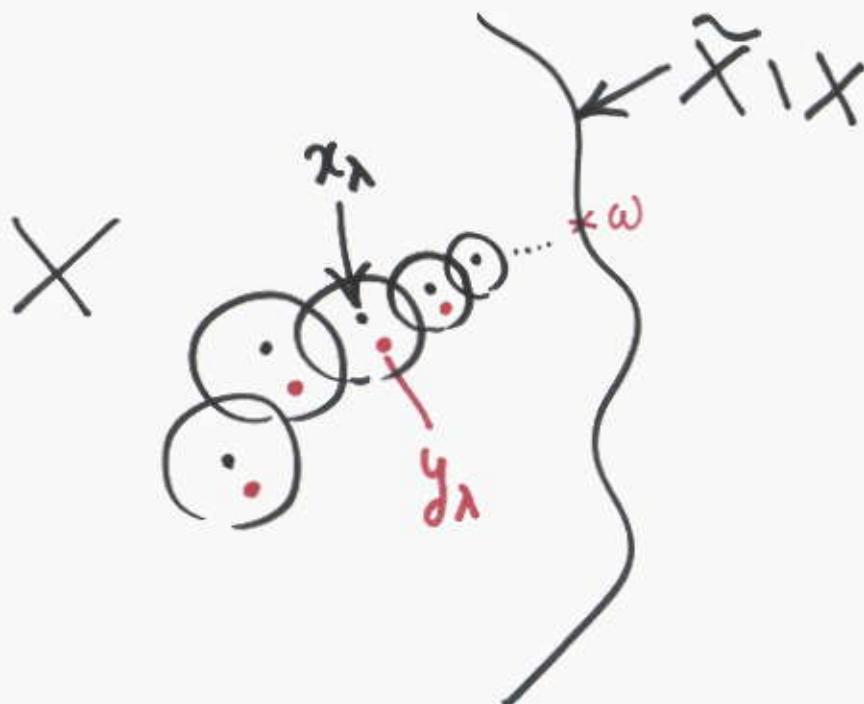
example 3 topological coarse structure

X locally compact T_2 space

$X \subset \tilde{X}$ compactification

$$\mathcal{E}_{\tilde{X}} = \left\{ E \subset X^2 : \begin{array}{l} (x_\lambda), (y_\lambda) : \text{nets in } X, (x_\lambda, y_\lambda) \in E, \\ x_\lambda \rightarrow \omega \in \tilde{X} \setminus X \Rightarrow y_\lambda \rightarrow \omega \end{array} \right\}$$

The topological coarse str. induced by \tilde{X} .



2.2 Higson coronas of coarse spaces.

Definition. X loc cpt T_2 sp with a coarse structure, where

(*) $B \subset X$ is bounded $\Leftrightarrow \overline{B}$ is compact.

(examples 1–3 satisfy (*) !)

$$C_h(X) = \left\{ g: X \rightarrow \mathbb{R} : \begin{array}{l} g: \text{bdd, conti and} \\ \forall E \in \mathcal{E} \quad \lim_{x \rightarrow \infty} \text{diam } g(E[x]) = 0 \end{array} \right\}$$

The algebra of Higson functions

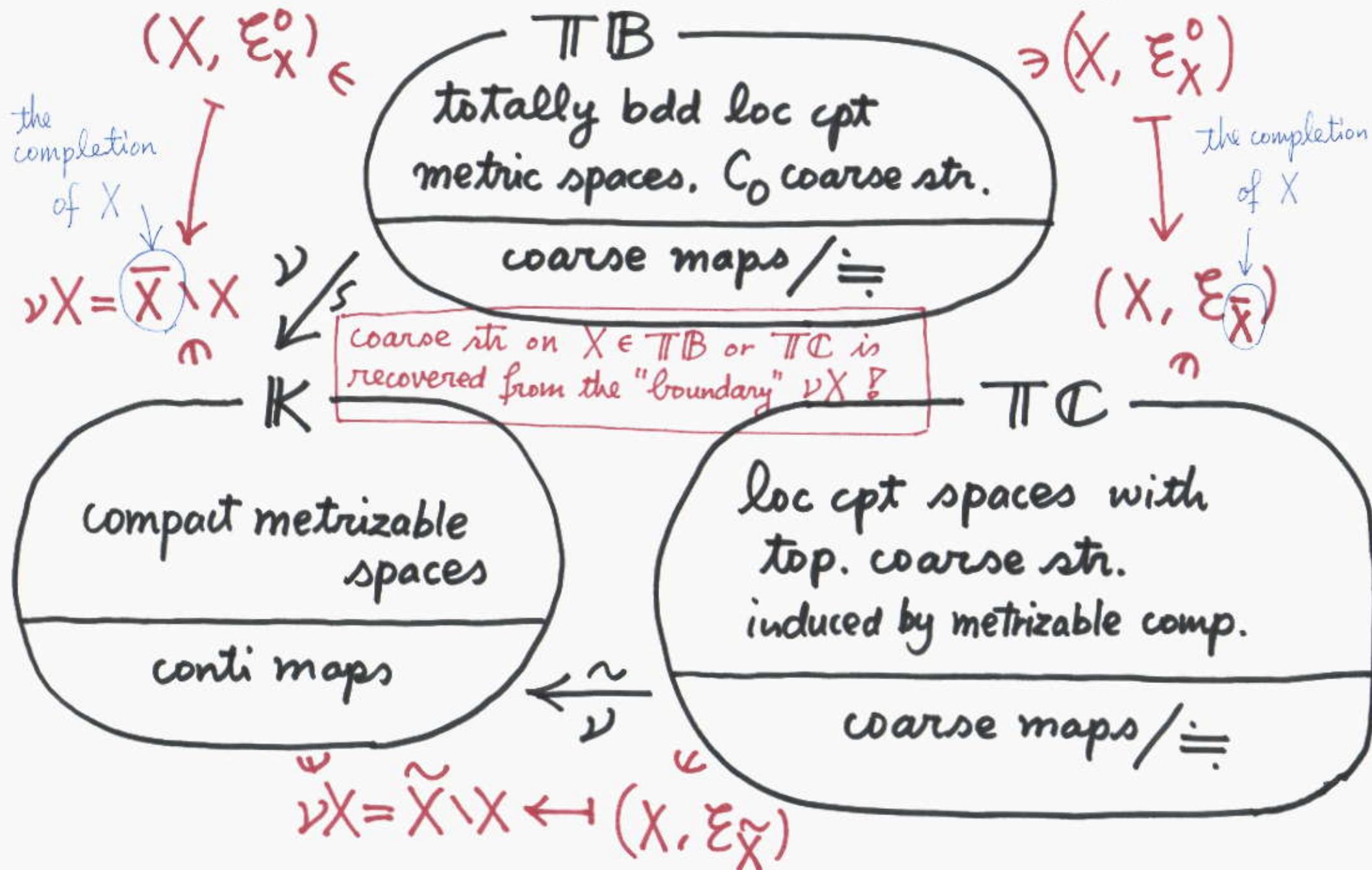
hX : the Higson compactification of X

$\nu X = hX \setminus X$: the Higson corona of X

• $f: X \rightarrow Y$ coarse map induces $\nu f: \nu X \rightarrow \nu Y$ conti.

$X \sim Y$ coarse eq. $\Rightarrow \nu X \approx \nu Y$ homeo

Theorem. (M-Y) There are equivalences of categories:



3. Related results.

(1) defining an operation on compacta

In general: $\begin{matrix} X, Y \text{ coarse sp} \\ (X, \mathcal{E}) \quad (Y, \mathcal{F}) \end{matrix} \rightsquigarrow \mathcal{E} \otimes \mathcal{F} := \left\{ G \subset (X \times Y)^2 : \begin{array}{l} \text{pr}_{X^2} G \in \mathcal{E}, \\ \text{pr}_{Y^2} G \in \mathcal{F} \end{array} \right\}$

is a coarse str on $X \times Y$
the product coarse str.

Let $K, L \in \mathbb{K}$ compact metrizable

$\xrightarrow[\text{Thm}]{\quad} \exists! (X, \mathcal{E}_X) \quad \exists! (Y, \mathcal{E}_Y) \in \Pi \mathbb{C} \text{ s.t. } \tilde{X} \setminus X = K, \tilde{Y} \setminus Y = L$

$$\nu(X, \mathcal{E}_X) \quad \nu(Y, \mathcal{E}_Y)$$

$\rightsquigarrow \tilde{X} \times \tilde{Y}$ is a metrizable comp. of $X \times Y$

Lemma. $\mathcal{E}_{\tilde{X} \times \tilde{Y}} \subset \mathcal{E}_X \otimes \mathcal{E}_Y$, hence

$\text{id}: (X \times Y, \mathcal{E}_{\tilde{X} \times \tilde{Y}}) \rightarrow (X \times Y, \mathcal{E}_X \otimes \mathcal{E}_Y) : \text{coarse map.}$

$\rightsquigarrow \nu \text{id}: \tilde{X} \times \tilde{Y} \setminus X \times Y \longrightarrow \nu(X \times Y, \mathcal{E}_X \otimes \mathcal{E}_Y).$ surj
contd

$\nu \text{id}: \tilde{X} \times \tilde{Y} \setminus X \times Y \rightarrow \nu(X \times Y, \mathcal{E}_{\tilde{X}} \otimes \mathcal{E}_{\tilde{Y}})$.



compact metrizable

$\therefore \nu(X \times Y, \mathcal{E}_{\tilde{X}} \otimes \mathcal{E}_{\tilde{Y}})$ is a compact metrizable sp
!! determined only by $K, L \in \mathbb{K}$.

$K \star L$.

Prop. \star is trivial: in fact,

$$K \star L = \begin{cases} \emptyset & \text{if } K = L = \emptyset \\ 1 \text{pt} & \text{otherwise.} \end{cases}$$

(2) when is $\nu f: \nu X \rightarrow \nu X$ fixed-point free?

$X = (X, \mathcal{E})$ coarse sp

$f: X \rightarrow X$ coarse map.

$\nu f: \nu X \rightarrow \nu X$ fixed-pt free \Leftrightarrow ?

Background (van Douwen 1993)

X : paracompact T_2 . $\dim X < \infty$

$\beta f: \beta X \rightarrow \beta X$ fixed-pt free \Leftrightarrow f : fixed-pt free

(X, \mathcal{E})

\downarrow coarse
version

Conj X : coarse space, $f: X \rightarrow X$ coarse map

$\nu f: \nu X \rightarrow \nu X$ fixed-pt free $\stackrel{\text{OK}}{\Leftrightarrow} \forall A \subset X \text{ unbounded } \{ (a, f(a)) ; a \in A \} \notin \mathcal{E}$

?

... (*)

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$$\begin{array}{c} \text{Conj. } \forall f: \nu X \rightarrow \nu X \\ \text{fixed-pt free} \end{array} \stackrel{\substack{\text{OK} \\ ?}}{\iff} \boxed{\begin{array}{l} \forall A \subset X \text{ unbounded} \\ \{(a, f(a)); a \in A\} \notin \Sigma \end{array}}$$

condition (*)

$$\begin{array}{c} \iff \\ X: \text{proper metric sp.} \quad \lim_{x \rightarrow \infty} d(x, f(x)) = \infty \\ \Sigma = \Sigma_X^b \end{array}$$

Prop Conj is true for :

- (1) $X \in \overline{\text{TC}}$ (top coarse str induced by metrizable comp.)
(in general, top coarse str induced by 1st ctable comp.)
- (2) X : Gromov hyperbolic geodesic space
(w/ bdd coarse str.)

(proof) (1) routine.

(2) Suppose (*) holds. Then $\lim_{x \rightarrow \infty} d(x, f(x)) = \infty$.

$\rightarrow \partial_\infty f: \partial_\infty X \rightarrow \partial_\infty X$ is fixed-pt free.

Fact If $g: X \rightarrow \mathbb{R}$ conti extends over $X \cup \partial_\infty X$,
then g is a Higson function.

$$\begin{array}{ccc} \hookrightarrow & X \cup vX & \xrightarrow{\exists \pi} X \cup \partial_\infty X \\ & \text{UI} & \text{UI} \\ & X = & X \end{array}$$

If $vf: vX \rightarrow vX$ has a fixed pt $\xi \in vX$,
then $\pi(\xi) \in \partial_\infty X$ would be a fixed pt of $\partial_\infty f$,
a contradiction. Therefore vf is fixed-pt free!

Q Is Conj true for \mathbb{R}^n ?

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metric space X with $\text{asdim } X < \infty$?
proper CAT(0) spaces ?