The large scale geometry

Žiga Virk

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The idea: to study the metric spaces from a large perspective. Compact sets $\approx$ points.
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\( f : X \rightarrow Y \) a map between metric spaces (no continuity required).

- \( f \) is **proper** if \( f^{-1}(A) \) is bounded for every bounded subset \( A \subset Y \);
- \( f \) is **bornologous** (or large scale uniform, or ls-uniform)
  \( \forall R < \infty, \exists S < \infty : \)
  \[
  d(x, y) < R \Rightarrow d(f(x), f(y)) < S
  \]
- \( f \) is **coarse** if it is proper and bornologous;
- \( f \) is **close** (or ls-equivalent) to \( g : Z \rightarrow Y \) if
  \( \exists R > 0 : d(f(x), g(x)) < R, \forall x \in X. \) [notation: \( f \sim_{ls} g \)]
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The idea

The coarse category

The group setting

Motivation

The combinatorial approach

Lipschitz partitions of unity

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Coarse category

- **Objects:** metric spaces;
- **Morphisms:** [equivalence classes of] coarse maps;
  - Spaces $X$ and $Y$ are coarsely equivalent $[X \sim_{ls} Y]$, if there exist coarse maps $f : X \to Y$ and $g : Y \to X$ so that $fg \sim_{ls} 1_Y$ and $gf \sim_{ls} 1_X$;
  - A special case of a bornologous map: $ls$-Lipschitz map (or $(c, A)$–Lipschitz map). They possess a linear bound on the size of the image:

$$d(f(x), f(y)) \leq cd(x, y) + A, \quad \forall x, y \in X$$

- $(c, A)$–bilipschitz maps:

$$c^{-1}d(x, y) - A \leq d(f(x), f(y)) \leq cd(x, y) + A, \quad \forall x, y \in X$$
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• Is-Lipschitz correspond with coarse maps in the usual setting of (coarsely) geodesic spaces;
  • \( \mathbb{R}^n \sim_{ls} \mathbb{Z}^n \);
  • \( X \sim_{ls} * \) iff \( X \) is bounded;
  • for every metric space \( X \) there exists a discrete \( Y \): \( X \sim_{ls} Y \).
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The group setting

\[ G = \langle g_1, \ldots, g_n \mid r_1, \ldots \rangle \] a finitely generated group.

Cayley graph \( \Gamma_G \):

- vertices: elements of \( G \);
- edges: \( [u, v] \iff \exists i : u = v g_i^{\pm 1} \);
- loops in \( \Gamma_G \) are "relations" on the generating set;
- \( \Gamma_G \) is geodesic;
- \( \Gamma_G \) depends on presentation, its coarse type does not.
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Lipschitz partitions of unity
Coarse invariants of groups

Gromov initiated intense research.

• being finite / finitely presented;
• virtually Abelian;
• virtually Nilpotent;
• virtually free;
• amenable;
• hyperbolic;
• asymptotic dimension;
• ends of group;
• growth rate.

Group $G$ has virtual property $P$ if there exists a finite index subgroup $H \leq G$ with property $P$. 
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Application to Novikov conjecture by Yu:

- Novikov conjecture holds for groups of finite asymptotic dimension;

- Novikov conjecture holds for groups with property A.

Important aspect: coarse embeddings into Hilbert space.
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Important aspect: coarse embeddings into Hilbert space.
Property A can be considered as a variant of the amenability.

A countable discrete group $G$ is **amenable** if $\forall R, \varepsilon > 0$ exists a finitely supported $f \in \ell^1(G)$ such that:

1. $\|f\|_1 = 1$;
2. $\|gf - f\|_1 < \varepsilon, \forall g \in B(e, R)$.

A countable discrete group $G$ has **property A** if there exists $1 \leq p < \infty$ so that $\forall R, \varepsilon > 0$ exists $F : G \rightarrow \ell^1(G)$ with the following properties:

1. $\|F_x\|_p = 1, \forall x \in G$;
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3. $\exists S > 0 : \text{supp } F_x \subset B(x, S), \forall x \in G$.

Free group on two elements has property A but is not amenable.
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Free group on two elements has property A but is not amenable.
Property A can be considered as a weakening of the finite asymptotic dimension.

A discrete metric space of bounded geometry has **property A**, if \( \forall R, \varepsilon > 0, \exists S > 0 \) and a family of finite nonempty subsets \( \{A_x\}_{x \in X} \) of \( X \times \mathbb{N} \) so that:

1. if \( x, y \in X, d(x, y) \leq R \) then \( \frac{|A_x \Delta A_y|}{|A_x \cap A_y|} < \varepsilon \);
2. if \( x \in X \) and \((y, n) \in A_x\) then \( d(x, y) \leq S \).

A discrete metric space of bounded geometry has **asymptotic dimension at most** \( n \) if, additionally, the projection of \( A_x \) onto \( X \) has at most \( n + 1 \) elements, \( \forall x \in X, \forall R, \varepsilon > 0 \).
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The combinatorial approach

Idea: dualization of the shape theory. Representation of the coarse structure by approximating complexes.

Conventions:

- short map: Lipschitz map with constant 1;
- \( A(\Gamma) \) for a graph \( \Gamma \): graph \( \Gamma \) with added edges. For vertices \( u, v, d(u, v) = 2 \) the edge \([u, v]\) is added;
- \( A(\Gamma) \) of a graph \( \Gamma \): \( A(\Gamma) \) is a complex of \( \Gamma \) so that \( \Delta \) is a simplex of \( A(\Gamma) \) whenever \([v, w]\) belongs to \( \Gamma \) for all \( v, w \in \Delta \);
- motivation: Rips graphs of a space.
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Approximation by the system of graphs

Objects: Coarse graphs.

A coarse graph is a direct sequence \( \{ V_1 \to V_2 \to \ldots \} \) of graphs \( V_n \) and short maps \( i_{n,m}: V_n \to V_m \) for all \( n \leq m \) such that

1. \( i_{n,n} = id \) for all \( n \geq 1 \),
2. \( i_{n,k} = i_{m,k} \circ i_{n,m} \) for all \( n \leq m \leq k \),
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Morphisms: equivalence classes of pre-morphisms.

Suppose $\mathcal{V} = \{ V_1 \overset{i_{1,2}}{\rightarrow} V_2 \overset{i_{2,3}}{\rightarrow} \cdots \}$ and $\mathcal{W} = \{ W_1 \overset{j_{1,2}}{\rightarrow} W_2 \overset{j_{2,3}}{\rightarrow} \cdots \}$ are two coarse graphs. A pre-morphism $F : \mathcal{V} \rightarrow \mathcal{W}$: a function $n_F : \mathbb{N} \rightarrow \mathbb{N}$, and short maps $f_k : V_k \rightarrow W_{n_F(k)}$ so that for every $k \geq 1$ there is $m \geq n_F(k+1)$ resulting in $j_{n_F(k),m} \circ f_k \sim_{ls} j_{n_F(k+1),m} \circ f_{k+1} \circ i_{k,k+1}$.
Two pre-morphisms $F, G : \mathcal{V} \to \mathcal{W}$ are considered to be equivalent if for every $k$ there is $m \geq \max\{n_F(k), n_G(k)\}$ so that $j_{n_F(k), m \circ f_k} \sim_{ls} j_{n_G(k), m \circ g_k}$. The sets of equivalence classes of pre-morphisms form the set of morphisms from $\mathcal{V}$ to $\mathcal{W}$.

Coarse graphs $\mathcal{V}$ and $\mathcal{W}$ are $ls$-equivalent if there exist pre-morphisms $F : \mathcal{V} \to \mathcal{W}$ and $G : \mathcal{W} \to \mathcal{V}$ such that $G \circ F$ is equivalent to $id_\mathcal{V}$ and $F \circ G$ is equivalent to $id_\mathcal{W}$. 
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The idea
The coarse category
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The combinatorial approach
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Two pre-morphisms \( F, G : \mathcal{V} \to \mathcal{W} \) are considered to be equivalent if for every \( k \) there is \( m \geq \max\{n_F(k), n_G(k)\} \) so that \( j_{n_F(k),m} \circ f_k \sim_{ls} j_{n_G(k),m} \circ g_k \). The sets of equivalence classes of pre-morphisms form the set of morphisms from \( \mathcal{V} \) to \( \mathcal{W} \).

Coarse graphs \( \mathcal{V} \) and \( \mathcal{W} \) are \textbf{ls-equivalent} if there exist pre-morphisms \( F : \mathcal{V} \to \mathcal{W} \) and \( G : \mathcal{W} \to \mathcal{V} \) such that \( G \circ F \) is equivalent to \( id_{\mathcal{V}} \) and \( F \circ G \) is equivalent to \( id_{\mathcal{W}} \).
Associating a coarse graph to the metric space.

- **Rips graphs** $RipsG_r$ for $r \to \infty$:
  1. maps $i$ are identity;
  2. $[u, v]$ is an edge in $RipsG_r$ iff $d(u, v) \leq r$.

- Rips graph of an increasing sequence of uniformly bounded covers $\mathcal{U}_n$ for which Lebesgue number $\to \infty$:
  1. maps $i$ are identity;
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If $\mathcal{V} = \{V_1 \to V_2 \to \ldots\}$ is a coarse graph of $(X, d_X)$ and $\mathcal{W} = \{W_1 \to W_2 \to \ldots\}$ is a coarse graph of $(Y, d_Y)$, then there is a natural bijection between bornologous maps from $X$ to $Y$ and morphisms from $\mathcal{V}$ to $\mathcal{W}$.

The description is sufficient for the formulation of the coarse type but not for some other invariants.
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Approximation by the system of complexes

Objects: Coarse complexes.

A **coarse simplicial complex** is a direct sequence \( \{ V_1 \to V_2 \to \ldots \} \) of simplicial complexes \( V_n \) and simplicial maps \( i_{n,m} : V_n \to V_m \) for all \( n \leq m \) such that

1. \( i_{n,n} = id \) for all \( n \geq 1 \),
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Suppose $\mathcal{V} = \{V_1 \to V_2 \to \ldots\}$ and $\mathcal{W} = \{W_1 \to W_2 \to \ldots\}$ are two coarse coarse simplicial complexes. A **pre-morphism** $F: \mathcal{V} \to \mathcal{W}$: a function $n_F: \mathbb{N} \to \mathbb{N}$, and simplicial maps $f_k: V_k \to W_{n_F(k)}$ so that for every $k \geq 1$ there is $m \geq n_F(k + 1)$ resulting in

$$j_{n_F(k),m} \circ f_k \sim ls j_{n_F(k+1),m} \circ f_{k+1} \circ i_{k,k+1}.$$
Associating a coarse simplicial complex to the metric space.

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Suppose $\mathcal{U}_n$ is a sequence of uniformly bounded covers of $X$ such that $\mathcal{U}_{n-1}$ is a star refinement of $\mathcal{U}_n$ for each $n \geq 1$. Then the sequence $\mathcal{N}(\mathcal{U}_1) \to \mathcal{N}(\mathcal{U}_2) \to \ldots$ of nerves of covers $\mathcal{U}_n$ forms a coarse simplicial complex if $i_{n,n+1}(U)$ contains the star $st(U,\mathcal{U}_n)$ for each $U \in \mathcal{U}_n$. Any such coarse complex will be denoted by Čech$_\star(X)$ and called a coarse Čech complex of $X$.

If $\mathcal{V} = \{V_1 \to V_2 \to \ldots\}$ is a coarse simplicial complex of $(X,d_X)$ and $\mathcal{W} = \{W_1 \to W_2 \to \ldots\}$ is a coarse simplicial complex of $(Y,d_Y)$, then there is a natural bijection between bornologous maps from $X$ to $Y$ and morphisms from $\mathcal{V}$ to $\mathcal{W}$. 
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Simplicial maps \( f, g : K \to L \) between simplicial complexes are **contiguous** if for every simplex \( \Delta \) of \( K \), \( f(\Delta) \cup g(\Delta) \) is contained in some simplex of \( L \).

Given a coarse simplicial complex \( K \) we say its **asymptotic dimension** is at most \( n \) (notation: \( asdim(K) \leq n \)) if for each \( m \) there is \( k > m \) such that \( i_{m,k} \) factors contiguously through an \( n \)-dimensional simplicial complex.
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\[
\cdots \to K_m \xrightarrow{i_{m,k}} \cdots \to K_k \to \cdots
\]

\[ L^{(n)} \]
A coarse simplicial complex $\mathcal{K} = \{ K_1 \to K_2 \to \ldots \}$ has Property A if for each $k \geq 1$ and each $\epsilon > 0$ there is $n > k$ and a function $f: |K_k|_m \to |K_n|_m$ such that $f$ is contiguous to $i_{k,n}: |K_k| \to |K_n|$ and the diameter of $f(\Delta)$ is at most $\epsilon$ for each simplex $\Delta$ of $K_k$. 

\[ i_{m,k} \]

\[ \cdots \to K_k \to \cdots \to K_n \to \cdots \]

\[ |K_k|_m \to |K_n|_m \]
Coarse simple connectivity

A metric space $X$ is **coarsely $k$-connected** if for each $r$ there exists $R \geq r$ so that the mapping $|Rips_r(X)| \to |Rips_R(X)|$ induces a trivial map of $\pi_i$ for $0 \leq i \leq k$.

1. A finitely generated group is coarsely 1-connected iff it is finitely presented;

2. [Fujiwara, White] Suppose $X$ is a geodesic metric space. $X$ is quasi-isometric to a simplicial tree if $H_1(X)$ is uniformly generated and $X$ is of asymptotic dimension 1;

3. An application of the previous item: finitely presented groups of asymptotic dimension 1 are virtually free;

4. If the fundamental group of a compact, connected, locally connected metric space is countable, then it is finitely presented.
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2. $X$ geodesic $\Rightarrow \tilde{X}$ geodesic;
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Partitions of unity

Given $\delta > 0$ and a simplicial complex $K$, $f : X \to K$ is a $\delta$–partition of unity if it is $(\delta, \delta)$–Lipschitz and $\{ f^{-1}(st(v)) \}_{v \in K(0)}$ is a uniformly bounded cover with Lebesgue number at least $\delta^{-1}$.

A metric space $X$ is large scale paracompact iff for every $\delta > 0$ there exists a $\delta$–partition of unity.

A metric space of bounded geometry is large scale paracompact iff it has property A.
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