The large scale geometry

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The idea: to study the metric spaces from a large perspective.

Compact sets $\approx$ points.

The idea

The coarse category

The group setting

Motivation

The combinatorial approach

Lipschitz partitions of unity

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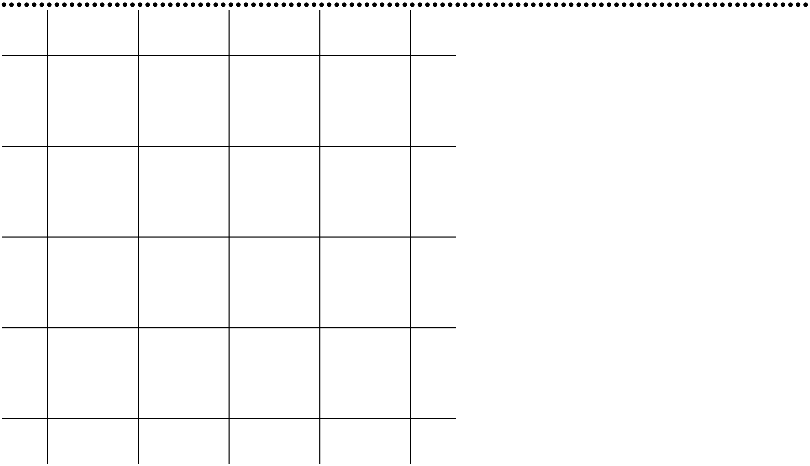

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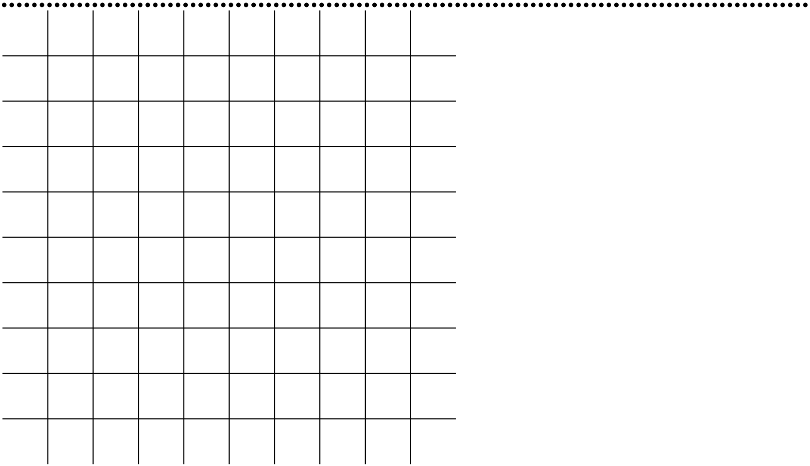

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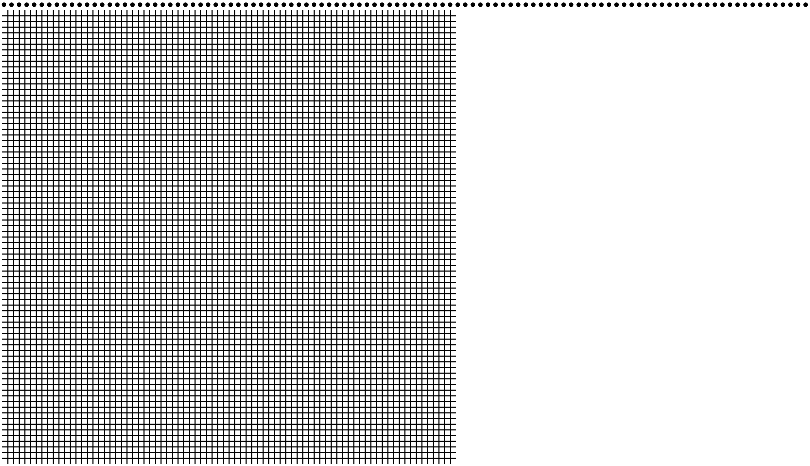

$f : X \to Y$ a map between metric spaces (no continuity required).

- f is proper if $f^{-1}(A)$ is bounded for every bounded subset $A \subset Y$;
- f is bornologous (or large scale uniform, or ls-uniform) $\forall R < \infty, \exists S < \infty$:

$$d(x, y) < R \Rightarrow d(f(x), f(y)) < S$$

- f is coarse if it is proper and bornologous;
- f is close (or ls-equivalent) to $g : Z \to Y$ if $\exists R > 0 : d(f(x), g(x)) < R, \forall x \in X$. [notation: $f \sim_{ls} g$]

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Coarse category

- Objects: metric spaces;
- Morphisms: [equivalence classes of] coarse maps;
- Spaces X and Y are coarsely equivalent $[X \sim_{ls} Y]$, if there exist coarse maps $f\colon X \to Y$ and $g\colon Y \to X$ so that $fg \sim_{ls} 1_Y$ and $gf \sim_{ls} 1_X$;
- A special case of a bornologous map: ls-Lipschitz map (or $(c, A)-$Lipschitz map). They possess a linear bound on the size of the image:

$$d(f(x), f(y)) \leq cd(x, y) + A, \quad \forall x, y \in X$$

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$$c^{-1}d(x, y) - A \leq d(f(x), f(y)) \leq cd(x, y) + A, \quad \forall x, y \in X$$

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- $\mathbb{R}^n \sim_{ls} \mathbb{Z}^n$;
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The group setting

$G =< g_1, \ldots, g_n \mid r_1, \ldots >$ a finitely generated group.
Cayley graph Γ_G:

- vertices: elements of G;
- edges: $[u, v] \Leftrightarrow \exists i : u = vg_i^{\pm 1}$;
- loops in Γ_G are "relations" on the generating set;
- Γ_G is geodesic;
- Γ_G depends on presentation, its coarse type does not.

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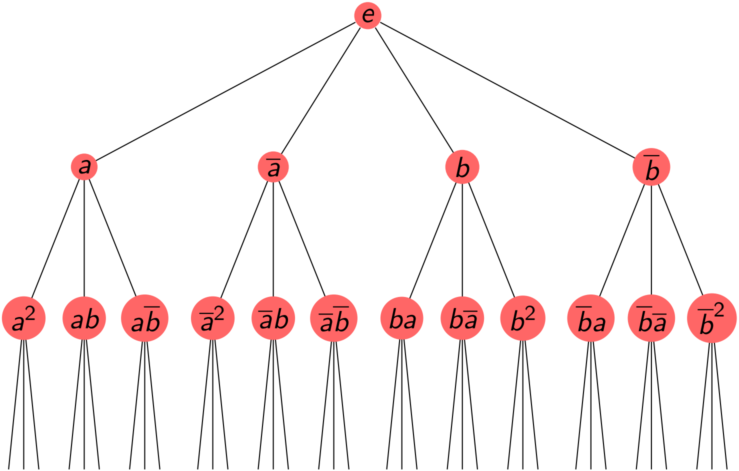
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Coarse invariants of groups

Gromov initiated intense research.

- being finite / finitely presented;
- virtually Abelian;
- virtually Nilpotent;
- virtually free;
- amenable;
- hyperbolic;
- asymptotic dimension;
- ends of group;
- growth rate.

Group G has **virtual property P** if there exists a finite index subgroup $H \leq G$ with property P.

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Important aspect: coarse embeddings into Hilbert space.

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Property A can be considered as a variant of the amenability.

A countable discrete group G is **amenable** if $\forall R, \varepsilon > 0$ exists a finitely supported $f \in \ell^1(G)$ such that:

1. $||f||_1 = 1$;
2. $||gf - f||_1 < \varepsilon, \forall g \in B(e, R)$.

A countable discrete group G has **property A** if there exists $1 \leq p < \infty$ so that $\forall R, \varepsilon > 0$ exists $F\colon G \to \ell^1(G)$ with the following properties:

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Free group on two elements has property A but is not amenable.

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Property A can be considered as a weakening of the finite asymptotic dimension.

A discrete metric space of bounded geometry has **property A**, if $\forall R, \varepsilon > 0, \exists S > 0$ and a family of finite nonempty subsets $\{A_x\}_{x \in X}$ of $X \times \mathbb{N}$ so that:

1. if $x, y \in X, d(x,y) \leq R$ then $\frac{|A_x \Delta A_y|}{|A_x \cap A_y|} < \varepsilon$;
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A discrete metric space of bounded geometry has **asymptotic dimension at most** n if, additionally, the projection of A_x onto X has at most $n+1$ elements, $\forall x \in X, \forall R, \varepsilon > 0$.

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The combinatorial approach

Idea: dualization of the shape theory. Representation of the coarse structure by approximating complexes.

Conventions:

- short map: Lipschitz map with constant 1;
- $A(\Gamma)$ for a graph Γ: graph Γ with added edges. For vertices $u, v, d(u, v) = 2$ the edge $[u, v]$ is added;
- $A(\Gamma)$ of a graph Γ: $A(\Gamma)$ is a complex of Γ so that Δ is a simplex of $A(\Gamma)$ whenever $[v, w]$ belongs to Γ for all $v, w \in \Delta$;
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Approximation by the system of graphs

Objects: Coarse graphs.

A **coarse graph** is a direct sequence $\{V_1 \to V_2 \to \dots\}$ of graphs V_n and short maps $i_{n,m} \colon V_n \to V_m$ for all $n \leq m$ such that

1. $i_{n,n} = id$ for all $n \geq 1$,
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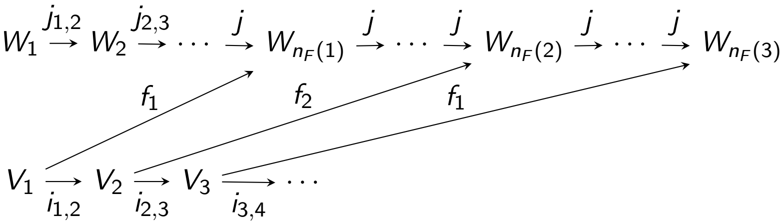
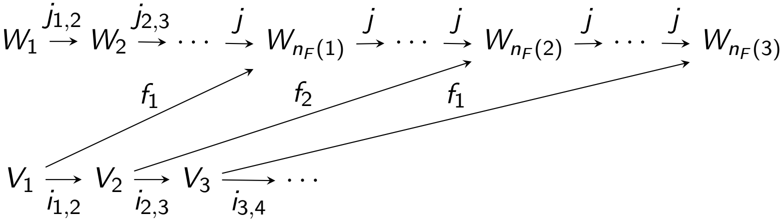
Morphisms: equivalence classes of pre-morphisms.

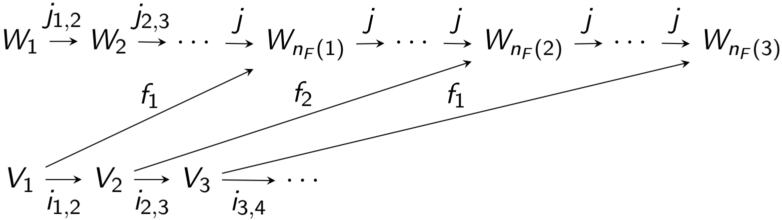
Suppose $\mathcal{V} = \{V_1 \stackrel{i_{1,2}}{\to} V_2 \stackrel{i_{2,3}}{\to} \dots\}$ and $\mathcal{W} = \{W_1 \stackrel{j_{1,2}}{\to} W_2 \stackrel{j_{2,3}}{\to} \dots\}$ are two coarse graphs. A **pre-morphism** $F\colon \mathcal{V} \to \mathcal{W}$: a function $n_F\colon \mathbb{N} \to \mathbb{N}$, and short maps $f_k\colon V_k \to W_{n_F(k)}$ so that for every $k \geq 1$ there is $m \geq n_F(k+1)$ resulting in $j_{n_F(k),m} \circ f_k \sim_{ls} j_{n_F(k+1),m} \circ f_{k+1} \circ i_{k,k+1}$.





Two pre-morphisms $F, G\colon \mathcal{V} \to \mathcal{W}$ are considered to be equivalent if for every k there is $m \geq \max\{n_F(k), n_G(k)\}$ so that $j_{n_F(k),m} \circ f_k \sim_{ls} j_{n_G(k),m} \circ g_k$. The sets of equivalence classes of pre-morphisms form the set of morphisms from $\mathcal{V}$ to $\mathcal{W}$.

Coarse graphs $\mathcal{V}$ and $\mathcal{W}$ are **ls-equivalent** if there exist pre-morphisms $F\colon \mathcal{V} \to \mathcal{W}$ and $G\colon \mathcal{W} \to \mathcal{V}$ such that $G \circ F$ is equivalent to $id_{\mathcal{V}}$ and $F \circ G$ is equivalent to $id_{\mathcal{W}}$.



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Associating a coarse graph to the metric space.

- Rips graphs $RipsG_r$ for $r \to \infty$:
  1. maps i are identity;
  2. $[u, v]$ is an edge in $RipsG_r$ iff $d(u, v) \leq r$.
- Rips graph of an increasing sequence of uniformly bounded covers $\mathcal{U}_n$ for which Lebesgue number $\to \infty$:
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If $\mathcal{V} = \{V_1 \to V_2 \to \dots\}$ is a coarse graph of (X, d_X) and $\mathcal{W} = \{W_1 \to W_2 \to \dots\}$ is a coarse graph of (Y, d_Y), then there is a natural bijection between bornologous maps from X to Y and morphisms from $\mathcal{V}$ to $\mathcal{W}$.

The description is sufficient for the formulation of the coarse type but not for some other invariants.

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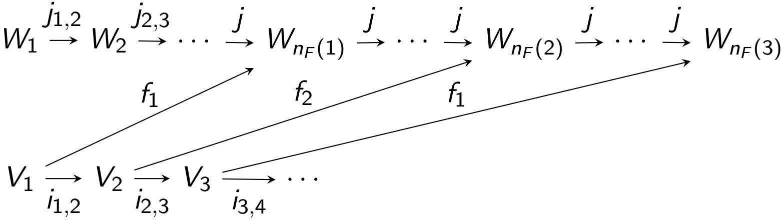
Lipschitz partitions of unity

Morphisms: equivalence classes of pre-morphisms.

Suppose $\mathcal{V} = \{V_1 \stackrel{i_{1,2}}{\to} V_2 \stackrel{i_{2,3}}{\to} \ldots\}$ and $\mathcal{W} = \{W_1 \stackrel{j_{1,2}}{\to} W_2 \stackrel{j_{2,3}}{\to} \ldots\}$ are two coarse coarse simplicial complexes. A **pre-morphism** $F \colon \mathcal{V} \to \mathcal{W}$: a function $n_F \colon \mathbb{N} \to \mathbb{N}$, and simplicial maps $f_k \colon V_k \to W_{n_F(k)}$ so that for every $k \geq 1$ there is $m \geq n_F(k+1)$ resulting in $j_{n_F(k),m} \circ f_k \sim_{ls} j_{n_F(k+1),m} \circ f_{k+1} \circ i_{k,k+1}$.



Associating a coarse simplicial complex to the metric space.

- Rips complexes $Rips_r$ for $r \to \infty$:
  1. maps i are identity;
  2. $[u_0, u_1, u_2, \dots, u_k]$ is a simplex in $Rips_r$ iff $Diam\{u_0, u_1, u_2, \dots, u_k\} \leq r$.
- Rips complex of an increasing sequence of uniformly bounded covers $\mathcal{U}_n$ for which Lebesgue number $\to \infty$:
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Suppose $\mathcal{U}_n$ is a sequence of uniformly bounded covers of X such that $\mathcal{U}_{n-1}$ is a star refinement of $\mathcal{U}_n$ for each $n \geq 1$. Then the sequence $\mathcal{N}(\mathcal{U}_1) \to \mathcal{N}(\mathcal{U}_2) \to \ldots$ of nerves of covers $\mathcal{U}_n$ forms a coarse simplicial complex if $i_{n,n+1}(U)$ contains the star $st(U, \mathcal{U}_n)$ for each $U \in \mathcal{U}_n$. Any such coarse complex will be denoted by $\check{\text{C}}\text{ech}_*(X)$ and called a **coarse Čech complex of X**.

If $\mathcal{V} = \{V_1 \to V_2 \to \ldots\}$ is a coarse simplicial complex of (X, d_X) and $\mathcal{W} = \{W_1 \to W_2 \to \ldots\}$ is a coarse simplicial complex of (Y, d_Y), then there is a natural bijection between bornologous maps from X to Y and morphisms from $\mathcal{V}$ to $\mathcal{W}$.

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Asymptotic dimension

Simplicial maps $f, g\colon K \to L$ between simplicial complexes are **contiguous** if for every simplex Δ of K, $f(\Delta) \cup g(\Delta)$ is contained in some simplex of L.

Given a coarse simplicial complex $\mathcal{K}$ we say its **asymptotic dimension** is at most n (notation: $asdim(\mathcal{K}) \leq n$) if for each m there is $k > m$ such that $i_{m,k}$ factors contiguously through an n-dimensional simplicial complex.

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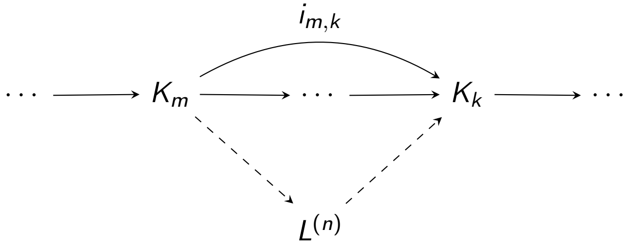

The idea

The coarse category

The group setting

Motivation

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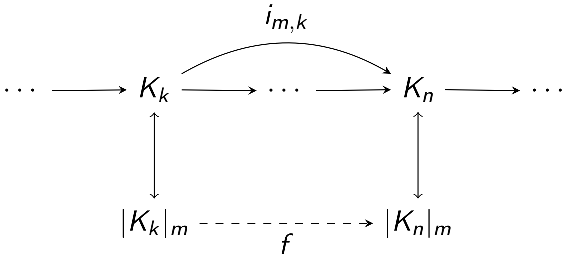
Property A

A coarse simplicial complex $\mathcal{K} = \{K_1 \to K_2 \to \dots\}$ has **Property A** if for each $k \geq 1$ and each $\epsilon > 0$ there is $n > k$ and a function $f \colon |K_k|_m \to |K_n|_m$ such that f is contiguous to $i_{k,n} \colon |K_k| \to |K_n|$ and the diameter of $f(|\Delta|)$ is at most ϵ for each simplex Δ of K_k.



Coarse simple connectivity

A metric space X is **coarsely k-connected** if for each r there exists $R \geq r$ so that the mapping $|Rips_r(X)| \to |Rips_R(X)|$ induces a trivial map of π_i for $0 \leq i \leq k$.

1. A finitely generated group is coarsely 1−connected iff it is finitely presented;
2. [Fujiwara, White] Suppose X is a geodesic metric space. X is quasi-isometric to a simplicial tree if $H_1(X)$ is uniformly generated and X is of asymptotic dimension 1;
3. An application of the previous item: finitely presented groups of asymptotic dimension 1 are virtually free;
4. If the fundamental group of a compact, connected, locally connected metric space is countable, then it is finitely presented.

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1. $\pi_1(X)$ countable $\Rightarrow \exists \widetilde{X}$;
2. X geodesic $\Rightarrow \widetilde{X}$ geodesic;
3. $\pi_1(X)$ acts on $\widetilde{X}$ $\Rightarrow$ [Švarc-Milnor Lemma] $\pi_1(X)$ finitely generated and $\pi_1(X) \simeq_{ls} \widetilde{X}$;
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Partitions of unity

Given $\delta > 0$ and a simplicial complex K, $f\colon X \to K$ is a $\delta-$**partition of unity** if it is $(\delta, \delta)-$Lipschitz and $\{f^{-1}(st(v))\}_{v \in K^{(0)}}$ is a uniformly bounded cover with Lebesgue number at least δ^{-1}.

A metric space X is **large scale paracompact** iff for every $\delta > 0$ there exists a $\delta-$partition of unity.

A metric space of bounded geometry is large scale paracompact iff it has property A.

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