Characters of countably tight spaces

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X: topological space, $A \subseteq X$

$\overline{A}$: closure of A

Definition 1. A space X is countably tight $\overset{\text{def.}}{\iff}$ For every $A \subseteq X$ and $x \in \overline{A}$, there is a countable $B \subseteq A$ with $x \in \overline{B}$.

In this talk, we will observe some connections between characters of countably tight spaces (of size $\aleph_1$) and large cardinals.

Topological games

Definition 2. X: space. $\mathcal{A}$, $\mathcal{B}$: families of sets.

α: ordinal.

X satisfies $G^{\alpha}(\mathcal{A},\mathcal{B})$ if Player ONE does not have a winning strategy in the following game of length α:

$$\begin{array}{r|cccccc} \text{ONE} & \mathcal{A}_0 & \mathcal{A}_1 & \cdots & \mathcal{A}_\xi & \cdots & (\mathcal{A}_\xi \in \mathcal{A}) \\ \hline \text{TWO} & B_0 & B_1 & \cdots & B_\xi & \cdots & (B_\xi \in \mathcal{A}_\xi) \end{array} \quad (\xi < \alpha)$$

TWO wins if $\{B_\xi : \xi < \alpha\} \in \mathcal{B}$, otherwise ONE wins.

A class of spaces which satisfy $G^{\omega}(\mathcal{A},\mathcal{B})$ are studied widely, but $G^{\kappa}(\mathcal{A},\mathcal{B})$ for $\kappa \geq \omega_1$ does not.

A space X is Lindelöf

$\overset{\text{def.}}{\iff}$ Every open cover of X has a countable subcover.

Definition 3 (Tall)**.** A space X is indestructibly Lindelöf if X is Lindelöf and satisfies $G^{\omega_1}(\mathcal{U},\mathcal{U})$, where $\mathcal{U}$ is the set of all open covers of X.

ONE	$\mathcal{U}_0$	$\mathcal{U}_1$	$\cdots$	$\mathcal{U}_\xi$	$\cdots$	($\mathcal{U}_\xi$: open cover)
TWO	O_0	O_1	$\cdots$	O_ξ	$\cdots$	($O_\xi \in \mathcal{U}_\xi$)

TWO wins $\iff$ $\{O_\xi : \xi < \omega_1\}$ is an open cover.

Fact 4 (Scheepers-Tall)**.** *A space X is indestructibly Lindelöf $\iff$ X is Lindelöf and every σ-closed forcing preserves the Lindelöfness of X.*

Examples

① Every Lindelöf space of size $\leq \aleph_1$ is indestructibly Lindelöf.

② Every second countable space is indestructibly Lindelöf (and a second countable T_2 space has size $\leq 2^{\aleph_0}$).

③ (Tall) $\{0,1\}^{\aleph_1}$ is compact, has size $2^{\aleph_1}$, and is not indestructibly Lindelöf.

Question 5. Is there an indestructibly Lindelöf space which has size $> 2^{\aleph_0}$.

Large cardinals and long game

Fact 6 (Dias-Tall, Tall-Usuba). *The following are equiconsistent:*

① $2^{\aleph_0} = \aleph_1$ *and there is no indestructibly Lindelöf* T_3 *space of weight* $\leq \aleph_1$ *and size* $> \aleph_1$.

② $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} > \aleph_2$, *and there is no Lindelöf* T_3 *space of weight* $\leq \aleph_1$ *and size* $\aleph_2$.

③ *There is an inaccessible cardinal.*

① One cannot prove "there is no indestructibly Lindelöf T_3 space of weight $\leq \aleph_1$ and size $> \aleph_1$".

② And (assuming the consistency of "$\exists$ inaccessible cardinal"), one cannot disprove "there is no indestructibly Lindelöf T_3 space of weight $\leq \aleph_1$ and size $> \aleph_1$".

Indestructible countable tightness

Definition 7 (Scheepers)**.** A space X is indestructibly countably tight if X is countably tight and satisfies $G^{\omega_1}(\Omega_x, \Omega_x)$ for every $x \in X$, where Ω_x is the set of all $A \subseteq X$ with $x \in \overline{A}$.

ONE	A_0	A_1	$\cdots$	A_ξ	$\cdots$	$(x \in \overline{A_\xi})$
TWO	x_0	x_1	$\cdots$	x_ξ	$\cdots$	$(x_\xi \in A_\xi)$

TWO wins $\iff$ $x \in \overline{\{x_\xi : \xi < \omega_1\}}$.

Fact 8 (Scheepers)**.** *A space X is indestructibly Lindelöf $\iff$ X is countably tight and ever σ-closed forcing preserves the countable tightness of X.*

Examples

① If X is countably tight and has character $\leq \aleph_1$, then X is indestructibly countably tight.

② Every countable space is indestructibly countably tight.

③ (Scheepers) $C_p(\{0,1\}^{\aleph_1})$ is countably tight, has size $2^{\aleph_1}$, character $2^{\aleph_1}$, and it is not indestructibly countably tight.

④ (Kada, M. Sakai) The sequential fun $S_{\aleph_1}$ is countably tight, has size $\aleph_1$, character $2^{\aleph_1}$, and it is not indestructibly countably tight.

Question 9. Is there a small indestructibly countably tight space of character $> \aleph_1$?

Non-CH case

Lemma 10. *For each* $\kappa \leq 2^{\aleph_0}$*, there is an indestructibly countably tight* $T_{3\frac{1}{2}}$ *space of size* $\aleph_1$ *and character* κ*.*

Proof. (M. Sakai) Fix a filter F over ω with character κ. Let X be the space $\omega + 1$ with the topology $\{\{n\} : n < \omega\} \cup \{A \cup \{\omega\} : A \in F\}$.

Let $Y = X \oplus D(\aleph_1)$. Then Y is an indestructibly Lindelöf $T_{3\frac{1}{2}}$ space of size $\aleph_1$ and character κ. □

Hence if $2^{\aleph_0} > \aleph_1$, there is an indestructibly countably tight $T_{3\frac{1}{2}}$ space of size $\aleph_1$ and character $\aleph_2$.

Main results

Theorem 11. *Let κ be an inaccessible cardinal. Then the following hold in $V^{\mathrm{Col}(\omega_1,<\kappa)}$:*

① $2^{\aleph_0} = \aleph_1$ *and* $2^{\aleph_1} = \aleph_2$.

② *There is no indestructibly countably tight space of size $\aleph_1$ and character $> \aleph_1$.*

③ *There is no indestructibly countably tight T_3 space of density $\leq \aleph_1$ and character $> \aleph_1$.*

(Assuming the consistency of $\exists$inaccessible cardinal,) One cannot prove the existence of an indestructibly countably tight space of size $\aleph_1$ and character $> \aleph_1$.

Main results

Theorem 12. *Let κ be an inaccessible cardinal. Then the following hold in $V^{\mathrm{Col}(\omega_1,<\kappa)\times Fn(\kappa^+,2,\omega_1)}$:*

① $2^{\aleph_0} = \aleph_1$ *and* $2^{\aleph_1} > \aleph_2$.

② *There is no countably tight space of size $\aleph_1$ and character $\aleph_2$.*

③ *There is no countably tight T_3 space of density $\leq \aleph_1$ and character $\aleph_2$.*

(Assuming the consistency of $\exists$inaccessible cardinal,) One cannot prove the existence of a countably tight space of size $\aleph_1$ and character $\aleph_2$.

Main results

Theorem 13. *Suppose* ω_2 *is not inaccessible in the constructible universe* L. *Then there is an indestructibly countably tight* $T_{3\frac{1}{2}}$ *space of size* $\aleph_1$ *and character* $\aleph_2$.

One cannot disprove the existence of a(n indestructibly) countably tight space of size $\aleph_1$ and character $\aleph_2$.

Main results

Corollary 14. *The following are equiconsistent:*

① *There is no indestructibly countably tight space of size* $\aleph_1$ *and character* $> \aleph_1$.

② *There is no countably tight space of size* $\aleph_1$ *and character* $\aleph_2$.

③ *There is an inaccessible cardinal.*

Hence one cannot prove and disprove the existence of a(n indestructibly) countably tight space of size $\aleph_1$ and character $\aleph_2$.

In this talk, we will sketch the proof of:

Proposition 15. *Suppose* $2^{\aleph_0} = \aleph_1$ *and there is a Kurepa tree which has just* κ *cofinal branches. Then there is an indestructibly countably tight* $T_{3\frac{1}{2}}$ *space of size* $\aleph_1$ *and character* κ*.*

Theorem 13 follows from combining this with:

Fact 16 (H. Sakai)**.** *If* ω_2 *is not inaccessible in the constructible universe* L*, then there is a Kurepa tree which has just* $\aleph_2$ *cofinal branches.*

Definition 17. X: space

$C_p(X)$ is the space of all continuous functions from X to $\mathbb{R}$ with pointwise convergence topology.

Note that $\chi(C_p(X)) = |X|$ and $|C_p(X)| \leq 2^{d(X)}$.

Fact 18 (Arkhangelskii-Pytkeev). *X: $T_{3\frac{1}{2}}$.*

Then the following are equivalent:

① *Each finite product of X is Lindelöf.*

② *$C_p(X)$ is countably tight.*

Lemma 19. *X: $T_{3\frac{1}{2}}$.*

Then the following are equivalent:

① *Each finite product of X is indestructibly Lindelöf.*

② *$C_p(X)$ is indestructibly countably tight.*

Proof. Repeat the proof of Arkhangelskii-Pytkeev theorem in generic extension. □

We will find a good indestructibly Lindelöf space X such that:

① $C_p(X)$ is indestructibly countably tight, and

② $C_p(X)$ has a subspace which has size $\aleph_1$ and character $> \aleph_1$.

A tree T is Kurepa if

① the height of T is ω_1,

② each level of T is countable,

③ T has strictly more than $\aleph_1$ cofinal branches.

Fact 20 (Dias-Tall). *Suppose there is a Kurepa tree T which has just κ cofinal branches. Then there is a indestructibly Lindelöf compact T_2 space of weight $\aleph_1$ and size* $\max\{2^{\aleph_0}, \kappa\}$.

That space is a branch space of T.

Lemma 21. *Let X be a compact space. If X is indestructibly Lindelöf, then each finite product of X is indestructibly Lindelöf.*

Hence if $2^{\aleph_0} = \aleph_1$ and there is a Kurepa tree which has just κ cofinal branches, then there is a compact T_2 space X of weight $\aleph_1$, size κ such that $C_p(X)$ is indestructibly countably tight.

Elementary submodel

Let θ be a large regular cardinal.

Lemma 22. *Suppose $2^{\aleph_0} = \aleph_1$. Suppose that there is a Kurepa tree which has just κ cofinal branch, and let X be Dias-Tall space obtained from T.*
Let $M \prec H(\theta)$ with $|M| = \aleph_1 \subseteq M$ and $X, T, C_p(X), \ldots \in M$. Then the subspace $M \cap C_p(X)$ of $C_p(X)$ has size $\aleph_1$ and character κ.

$M \cap C_p(X) \approx$ a subspace of $C_p(X)$ of size $\aleph_1$ which is closed under all definable functions from X to $H(\theta)$ without parameters.

$M \cap C_p(X)$ reflects some properties of $C_p(X)$, hence its character is κ.

Definition 23. X has countable fan tightness $\overset{\text{def.}}{\Longleftrightarrow}$ For every $x \in X$ and A_n $(n < \omega)$ with $x \in \overline{A_n}$, there are finite $B_n \subseteq A_n$ $(n < \omega)$ such that $x \in \overline{\bigcup_{n<\omega} B_n}$.

X is Menger $\overset{\text{def.}}{\Longleftrightarrow}$ For every open covers $\mathcal{U}_n$ $(n < \omega)$, there are finite $\mathcal{V}_n \subseteq \mathcal{U}_n$ such that $X = \bigcup_{n<\omega} \bigcup \mathcal{V}_n$.

countable fan tightness $\Rightarrow$ countable tightness.

compact $\Rightarrow$ Menger $\Rightarrow$ Lindelöf.

Fact 24 (Arkhangelskii). X: $T_{3\frac{1}{2}}$.
Then the following are equivalent.

① *Each finite product of* X *is Menger.*

② $C_p(X)$ *has countable fan tightness.*

Proposition 25. *There is a $T_{3\frac{1}{2}}$ space X of size $\aleph_1$ and character $2^{\aleph_1}$ such that X has countable fan tightness, but is not indestructibly countably tight.*

Proof. Since $\{0,1\}^{\aleph_1}$ is a compact T_2 space, $C_p(\{0,1\}^{\aleph_1})$ has countable fan tightness.

Take a large regular θ and take $M \prec H(\theta)$ with $|M| = \aleph_1 \subseteq M$. Then $M \cap C_p(X)$ has size $\aleph_1$, character $2^{\aleph_1}$, has countable fan tightness, but is not indestructibly countably tight. $\square$

ご清聴ありがとうございました.