ペアノ空間から１次元局所コンパクトＡＲへの
関数空間のコンパクト化

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Outline

0 Introduction

Spaces are regular and maps are continuous.
For spaces X and Y, $\mathrm{C}(X, Y)$ is the space of all maps from X to Y with the compact-open topology, which is generated by the following sets:

$$\{f \in \mathrm{C}(X, Y) \mid f(K) \subset U\},$$

where K is a compact set in X and U is an open set in Y.
When X is locally compact and σ-compact, and Y is metrizable, $\mathrm{C}(X, Y)$ is metrizable.
Let $\mathbf{Q} = [-1, 1]^{\mathbb{N}}$ be the Hilbert cube and $\mathbf{s} = (-1, 1)^{\mathbb{N}}$ be the pseudo-interior of $\mathbf{Q}$.

Main Theorem

Let X be an infinite, locally compact, locally connected, separable metrizable space and let Y be a 1-dimensional locally compact AR.
If X is non-discrete or Y is non-compact, then $\mathrm{C}(X,Y)$ has a natural compactification $\overline{\mathrm{C}}(X,Y)$ such that

$$(\overline{\mathrm{C}}(X,Y), \mathrm{C}(X,Y)) \approx (\mathbf{Q}, \mathbf{s}).$$

Remark

If X is discrete and Y is compact, then

$$\mathrm{C}(X,Y) \approx \mathbf{Q}.$$

1 Background

Let $\mathrm{Cld}(X)$ **be the set of all non-empty closed subsets of a space** X **and** $\mathrm{Cld}^*(X) = \mathrm{Cld}(X) \cup \{\emptyset\}$**.**
For each $Z \subset X$**, let**

$$Z^- = \{A \in \mathrm{Cld}^*(X) \mid A \cap Z \neq \emptyset\} \textbf{ and}$$
$$Z^+ = \{A \in \mathrm{Cld}^*(X) \mid A \subset Z\}.$$

By $\mathrm{Cld}^*_F(X)$**, we denote** $\mathrm{Cld}^*(X)$ **with the Fell topology, which is generated by the following sets:**

$$(X \setminus K)^+, \; U^-,$$

where K **is a compact set in** X **and** U **is an open set in** X**.**

$\mathrm{Cld}_F^*(X)$ **is a compact metrizable space if and only if** X **is a locally compact separable metrizable space.**

For each compact metric space $X = (X, d)$**, the relative topology on** $\mathrm{Cld}(X) \subset \mathrm{Cld}_F^*(X)$ **is induced by the Hausdorff metric** d_H **of** d**.**

When X **is a locally compact, locally connected space, and** Y **is a locally compact space,** $\mathrm{C}(X, Y)$ **can be regarded as a subspace of** $\mathrm{Cld}_F^*(X \times Y)$**, where each** $f \in \mathrm{C}(X, Y)$ **is identified with the graph of** f **in** $X \times Y$**.**

$\mathrm{cl}_{\mathrm{Cld}_F^*(X \times Y)} \mathrm{C}(X, Y)$ **is a natural metrizable compactification of** $\mathrm{C}(X, Y)$ **under the assumption of the main theorem.**

Theorem 1.1 [K. Sakai and S. Uehara (1999)
- A. Kogasaka and K. Sakai (2009)]

Let X be an infinite, locally compact, locally connected, separable metrizable space.
Then

$$(\mathrm{cl}_{\mathrm{Cld}_F^*(X\times\overline{\mathbb{R}})}\mathrm{C}(X,\mathbb{R}),\mathrm{C}(X,\mathbb{R}))\approx(\mathbf{Q},\mathbf{s}),$$

where $\overline{\mathbb{R}}=[-\infty,+\infty]$ is the extended real line.

Remark

$\mathrm{C}(\mathbf{I},\mathbb{R})$ **is not homotopy dense in** $\mathrm{cl}_{\mathrm{Cld}_F^*(\mathbf{I}\times\mathbb{R})}\mathrm{C}(\mathbf{I},\mathbb{R})$ **and** $\mathrm{cl}_{\mathrm{Cld}_F^*(\mathbf{I}\times\alpha\mathbb{R})}\mathrm{C}(\mathbf{I},\mathbb{R})$**, where $\alpha\mathbb{R}$ is the one point compactification of $\mathbb{R}$.**

2 Dendrites

Definition (Dendrite)

A dendrite is a Peano continuum containing no simple closed curves, equivalently it is a 1**-dimensional compact AR.**

Definition (Convex metric)

For a metric space $X = (X, d)$**,** d **is convex if for each** x**,** $y \in X$**, there exists** $z \in X$ **such that** $d(x, z) = d(y, z) = d(x, y)/2$**. When** d **is complete, there exists an arc from** x **to** y **isometric to the segment** $[0, d(x, y)]$**.**

Proposition 2.1

Every Peano continuum admits a convex metric. Hence every dendrite does so.

Proposition 2.2

For each dendrite D, there exists a map $\gamma : D^2 \times \mathbf{I} \to D$ such that for any distinct points x, $y \in D$, $\gamma(x, y, *) : \mathbf{I} \ni t \mapsto \gamma(x, y, t) \in D$ is the unique arc from x to y.

Proposition 2.3

Let D be a dendrite with E the end points. Then $D \setminus E$ is homotopy dense in D.

(Proof) Fix $x_0 \in D \setminus E$**, and define a homotopy** $h : D \times \mathbf{I} \to D$ **by** $h(x,t) = \gamma(x, x_0, t)$**, where** $\gamma : D^2 \times \mathbf{I} \to D$ **as in Proposition 2.2.** $\square$

Theorem 2.4

A space Y is a 1-dimensional locally compact AR if and only if Y has a dendrite compactification $\widetilde{Y}$ such that the remainder $\widetilde{Y} \setminus Y$ is closed and contained in the set of all end points of $\widetilde{Y}$.

(Proof) Use the following Curtis' result. $\square$

Theorem 2.5 [D.W. Curtis (1980)]

Every locally compact, connected, locally connected, metrizable space Y has a Peano compactification $\widetilde{Y}$ such that

($*$) for each non-empty connected open set U in $\widetilde{Y}$, the subset $U \cap Y$ is a non-empty connected set.

From now on let X be an infinite, locally compact, locally connected, separable metrizable space and Y a 1-dimensional locally compact AR, and fix a dendrite compactification $\widetilde{Y}$ of Y with the remainder $\widetilde{Y} \setminus Y$ closed in $\widetilde{Y}$ and consisting end points.
By Proposition 2.3, Y is homotopy dense in $\widetilde{Y}$.

Proposition 2.6
$\mathrm{C}(X, Y)$ **is homotopy dense in** $\mathrm{C}(X, \widetilde{Y})$**.**

Let

$$\overline{\mathrm{C}}(X, Y) = \mathrm{cl}_{\mathrm{Cld}^*_F(X \times \widetilde{Y})} \mathrm{C}(X, Y) = \mathrm{cl}_{\mathrm{Cld}^*_F(X \times \widetilde{Y})} \mathrm{C}(X, \widetilde{Y}).$$

3 The closure of $\mathrm{C}(X, Y)$ in $\mathrm{Cld}^*_F(X \times \widetilde{Y})$

For spaces W and Z, let

$$\mathrm{USCC}(W, Z) = \left\{ \phi : W \to \mathrm{Cld}(Z) \;\middle|\; \begin{array}{l} \phi \textbf{ is u.s.c. and} \\ \phi(w) \textbf{ is connected for every } w \in W\textbf{.} \end{array} \right\}$$

Identifying each $\phi \in \mathrm{USCC}(W, Z)$ with the graph of ϕ, we can regard $\mathrm{USCC}(W, Z) \subset \mathrm{Cld}^*(W \times Z)$.

Theorem 3.1

For each locally compact, locally connected, paracompact space W with no isolated points and each dendrite D,

$$\mathrm{cl}_{\mathrm{Cld}^*_F(W \times D)} \mathrm{C}(W, D) = \mathrm{USCC}(W, D).$$

Lemma 3.2

Let W be a locally compact, locally connected space and let Z be a compact connected space.
Then $\mathrm{USCC}(W, Z)$ is closed in $\mathrm{Cld}^*_F(W \times Z)$.

Lemma 3.3

Let W be a paracompact space with no isolated points and let D be a dendrite.
Then $\mathrm{C}(W, D)$ is dense in $\mathrm{USCC}(W, D)$.

(Proof) Use the following Michael's selection theorem. $\square$

Theorem 3.4 [E. Michael (1959)]

Let W be a paracompact space and D a dendrite.
For every l.s.c. set-valued function $\phi : W \to \mathrm{Cld}(D)$, if each $\phi(w)$ is connected, then ϕ has a continuous selection.

Hence if X is connected, then

$$\overline{\mathrm{C}}(X,Y) = \mathrm{cl}_{\mathrm{Cld}^*_F(X\times\widetilde{Y})}\,\mathrm{C}(X,Y) = \mathrm{cl}_{\mathrm{Cld}^*_F(X\times\widetilde{Y})}\,\mathrm{C}(X,\widetilde{Y})$$
$$= \mathrm{USCC}(X,\widetilde{Y}).$$

4 Proof of the Main Theorem

Theorem 4.1 [R.D. Anderson]
Let $M \subset \mathbf{Q}$. The pair $(\mathbf{Q}, M)$ is homeomorphic to $(\mathbf{Q}, \mathbf{Q} \setminus \mathbf{s})$ if and only if M is a cap set in $\mathbf{Q}$, that is, it is a Z_σ-set and has the following property:

(cap) **For each pair A, B of compact sets in $\mathbf{Q}$ with $B \subset A \cap M$ and each $\epsilon > 0$, there exists an embedding $h : A \to M$ such that $h|_B = \mathrm{id}_B$ and $d(h(a), a) < \epsilon$ for every $a \in A$, where d is an admissible metric for $\mathbf{Q}$.**

We show the following:

(1) $\mathrm{C}(X, Y) \approx$ **s.**

(2) $\mathrm{C}(X, Y)$ **is homotopy dense in** $\overline{\mathrm{C}}(X, Y)$**.**

(3) $\overline{\mathrm{C}}(X, Y) \approx$ **Q.**

(4) $\overline{\mathrm{C}}(X, Y) \setminus \mathrm{C}(X, Y)$ **is a cap set in** $\overline{\mathrm{C}}(X, Y)$**.**

(Case I) X **is discrete.**

When X **is discrete (so** Y **is non-compact),**

$$
\begin{aligned}
(\overline{\mathrm{C}}(X,Y), \mathrm{C}(X,Y)) &= (\mathrm{cl}_{\mathrm{Cld}^*_F(X\times\widetilde{Y})}\, \mathrm{C}(X,\widetilde{Y}), \mathrm{C}(X,Y)) \\
&= (\mathrm{C}(X,\widetilde{Y}), \mathrm{C}(X,Y)) \\
&\approx (\widetilde{Y}^{\mathbb{N}}, Y^{\mathbb{N}}).
\end{aligned}
$$

Theorem 4.2

Let D **be a dendrite and** E_0 **be a non-empty closed set of** D **which consists of end points.**

Then

$$(D^{\mathbb{N}}, (D \setminus E_0)^{\mathbb{N}}) \approx (\mathbf{Q}, \mathbf{s}).$$

(Case II) X **is non-discrete.**

Lemma 4.3

Let W_n **be a compact AR and** Z_n **be a homotopy dense** G_δ **subset of** W_n**,** $n \in \mathbb{N}$**.**

Then

$$(\mathbf{Q} \times \prod_{n\in\mathbb{N}} W_n, \mathbf{s} \times \prod_{n\in\mathbb{N}} Z_n) \approx (\mathbf{Q}, \mathbf{s}).$$

Hence it is sufficient to show the case X **is connected.**

First, we consider X **is compact.**

(2) $\mathrm{C}(X, Y)$ **is homotopy dense in** $\overline{\mathrm{C}}(X, Y)$**.**

By Proposition 2.6, it remains to prove that $\mathrm{C}(X, \widetilde{Y})$ **is homotopy dense in** $\mathrm{cl}_{\mathrm{Cld}^*_F(X \times \widetilde{Y})} \mathrm{C}(X, \widetilde{Y})$**.**

Theorem 4.4

For each non-degenerate Peano continuum W **and each dendrite** D**,** $\mathrm{C}(W, D)$ **is homotopy dense in** $\mathrm{cl}_{\mathrm{Cld}^*_F(W \times D)} \mathrm{C}(W, D)$**.**

Lemma 4.5 [K. Sakai and S. Uehara (1999)]

Let $W = (W, d)$ **be a compact metric space and** Z **be a dense subset of** W **which has the following property:**

(∗) There exists $\alpha \geq 1$ **such that for any locally finite simplicial complex** K**, each map** $f : K^{(0)} \to Z$ **extends to a map** $\tilde{f} : |K| \to Z$ **such that**

$$\operatorname{diam}_d \tilde{f}(\sigma) \leq \alpha \operatorname{diam}_d f(\sigma^{(0)}) \text{ for every } \sigma \in K.$$

Then Z **is homotopy dense in** W**.**

Let $W = (W, d_W)$ be a Peano continuum with a convex metric and $D = (D, d_D)$ be a dendrite with a convex metric. Define an admissible metric ρ on $W \times D$ as follows:

$$\rho((w_1, y_1), (w_2, y_2)) = \max\{d_W(w_1, w_2), d_D(y_1, y_2)\}$$

and denote by ρ_H the Hausdorff metric on $\mathrm{Cld}(W \times D)$ induced from it.

Lemma 4.6

Let K be a locally finite simplicial complex. If W is non-degenerate, then any map $f : K^{(0)} \to \mathrm{C}(W, D)$ extends to a map $\tilde{f} : |K| \to \mathrm{C}(W, D)$ such that

(*) $\mathrm{diam}_{\rho_H} \tilde{f}(\sigma) \leq 4 \, \mathrm{diam}_{\rho_H} f(\sigma^{(0)})$ **for each $\sigma \in K$.**

(4) $\overline{\mathrm{C}}(X,Y) \setminus \mathrm{C}(X,Y)$ **is a cap set in** $\overline{\mathrm{C}}(X,Y)$**.**

Theorem 4.7
$M = \mathrm{USCC}(X,\widetilde{Y}) \setminus \mathrm{C}(X,Y)$ **is a cap set in** $\mathrm{USCC}(X,\widetilde{Y})$**.**

(Proof) Take an admissible metric d_X **and an admissible convex metric** $d_{\widetilde{Y}}$ **on** X **and** $\widetilde{Y}$**, respectively, and define an admissible metric** ρ **on** $X \times \widetilde{Y}$ **as follows:**

$$\rho((x,y),(x',y')) = \max\{d_X(x,x'), d_{\widetilde{Y}}(y,y')\}.$$

By Theorem 4.1, it remains to show the following:

(cap) For each compacta $A, B \subset \mathrm{USCC}(X,\widetilde{Y})$ **with** $B \subset A \cap M$ **and each** $\epsilon > 0$**, there exists an embedding** $\tilde{h} : A \to M$

such that $\tilde{h}|_B = \mathrm{id}_B$ **and** $\rho_H(\tilde{h}(a), a) < \epsilon$ **for every** $a \in A$**, where** ρ_H **is the Hausdorff metric of** ρ**.**

Let $\alpha : A \to \mathbf{I}$ **be a map defined by** $\alpha(a) = \min\{1, \epsilon, \rho_H(a, B)\}/3$**.** **Since** $\mathrm{C}(X, Y)$ **is homotopy dense in** $\mathrm{USCC}(X, \widetilde{Y})$**, we can construct a map** $f : A \to \mathrm{USCC}(X, \widetilde{Y})$ **such that** $f|_B = \mathrm{id}_B$**,** $f(A \setminus B) \subset \mathrm{C}(X, Y)$ **and** $\rho_H(f(a), a) \le \alpha(a)$ **for every** $a \in A$**.**
In addition, we can find an embedding $g : A \setminus B \to \mathrm{C}(X, Y)$ **so that** $\rho_H(g(a), f(a)) < \alpha(a)$ **for each** $a \in A \setminus B$ **because** $\mathrm{C}(X, Y) \approx \mathbf{s}$**.**
Fix $x_0 \in X$ **and define** $h : A \setminus B \to \mathrm{Cld}(X \times \widetilde{Y})$ **by**

$$h(a)(x) = \begin{cases} \overline{B}(g(a)(x_0), \alpha(a)) & \textbf{if } x = x_0\textbf{,} \\ g(a)(x) & \textbf{if } x \neq x_0\textbf{,} \end{cases}$$

where $\overline{B}(g(a)(x_0), \alpha(a))$ **is the closed ball.**

Then each $h(a)$ **is an u.s.c. set-valued function.**
Because $d_{\widetilde{Y}}$ **is convex,** h **is continuous and** $\overline{B}(g(a)(x_0), \alpha(a))$ **is a non-degenerate subcontinuum of** $\widetilde{Y}$**.**
Hence $h(A \setminus B) \subset \mathrm{USCC}(X, \widetilde{Y}) \setminus \mathrm{C}(X, Y) = M$**.**
Since x_0 **is not isolated point and** g **is an injection,** h **is also an injection.**
It follows that

$$\begin{aligned}\rho_H(h(a), a) &\leq \rho_H(h(a), g(a)) + \rho_H(g(a), f(a)) + \rho_H(f(a), a) \\ &< 3\alpha(a) \leq \min\{1, \epsilon, \rho_H(a, B)\}.\end{aligned}$$

Therefore $h : A \setminus B \to M$ **can be extended to** $\tilde{h} : A \to M$ **by** $\tilde{h}|_B = \mathrm{id}_B$**.**
Since $h(A \setminus B) \cap B = \emptyset$**,** $\tilde{h}$ **is the desired embedding.** $\Box$

Next, we consider X is non-compact.
Take the one-point compactification $\alpha X = X \cup \{\infty\}$ of X, which is a Peano continuum.
Then use the result of the compact case.