Periodic points of maps and compactifications of spaces

Hisao Kato
University of Tsukuba

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Abstract. In this talk, we investigate some properties concerning periodic points of maps and compactifications of spaces. By use of the properties, we prove the following theorems:
(i) Let X be a finite-dimensional separable metric space and let $f : X \to X$ be a fixed-point free closed map with zero-dimensional set of periodic points. If $f : X \to X$ satisfies the condition

$$\sup\{|f^{-1}(x)|;\ x \in X\} < \infty,$$

then f is eventually 2-colorable.
(ii) Let X be a locally compact, separable metric finite-dimensional space. If $f : X \to X$ is any fixed-point free map with zero-dimensional set of periodic points, then f is eventually 2-colorable.

Let $f : X \to X$ be a map. Then f is a *finite-to-one* map if $|f^{-1}(x)| < \infty$ for each $x \in X$, where $|A|$ denotes the cardinal number of a set A. Also, we define ord(f) by the cardinal number $\sup_{x \in X} |f^{-1}(x)|$. Clearly ord($f$) $< \infty$ implies that f is finite-to-one. van Douwen proved that if $f : X \to X$ is a fixed-point free closed map of a finite-dimensional (separable) metric space X with ord(f) $< \infty$, then f is colorable and hence $\beta f : \beta X \to \beta X$ is fixed-point free, where βX denotes the Stone-Čech compactification of the space X. Buzyakova and Chigogidze proved that any fixed-point free map $f : X \to X$ of a locally compact and paracompact space X with $\dim X < \infty$, then f is colorable. Also, they showed that if $f : X \to X$ is any map of a locally compact and paracompact space X with $\dim X < \infty$, then Fix(βf) $= Cl_{\beta X}$Fix(f). We proved the following results:

(i) If X is a finite-dimensional separable metric space and $f : X \to X$ is a fixed-point free homeomorphism with zero-dimensional set of periodic points, then f is eventually 2-colorable.
(ii) Moreover, if X is a finite-dimensional compact metric space and $f : X \to X$ is any fixed-point free map with zero-dimensional set of periodic points, then f is eventually 2-colorable.

On the other hand, it is known that there exists a finite-to-one closed map $f : X \to X$ (Mazur's example) of a zero-dimensional separable metric space (in fact, the space of irrational numbers) X with no periodic point such that f is not eventually colorable, and hence we know that if $\gamma f : \gamma X \to \gamma X$ is any extension of f over any compactificatin γX of X, then γf has a fixed point.

In this talk, first, we investigate some properties concerning periodic points of maps and (Wallman) compactifications of spaces. Next, in view of the Mazur's example, we strengthen the above results (i) and (ii) to the following results (i') and (ii'):

(i') Let $f : X \to X$ be a fixed-point free closed map of a finite-dimensional separable metric space X with zero-dimensional set of periodic points. If $f : X \to X$ satisfies the condition ord$(f) < \infty$, then f is eventually 2-colorable.

(ii') Let X be a locally compact, separable metric finite-dimensional space. If $f : X \to X$ is any fixed-point free map with zero-dimensional set of periodic points, then f is eventually 2-colorable.

By the Mazur's example, in the statement of (i') we can not replace the condition "ord$(f) < \infty$" by the condition "finite-to-one map". Also, in the statement of (ii') we can not omit the condition "locally compact".

Let $\mathbb{N}$ be the set of all natural numbers, i.e., $\mathbb{N} = \{1, 2, ...\}$. For a map $f : X \to X$, let $P(f)$ be the set of all periodic points of f, i.e.,

$$P(f) = \{x \in X| \ f^p(x) = x \text{ for some } p \in \mathbb{N}\}.$$

For each $i \in \mathbb{N}$, we put $P_i(f) = \{x \in X| \ f^i(x) = x\}$(=Fix($f^i$)). Note that $P(f) = \bigcup\{P_i(f)| \ i \in \mathbb{N}\}$. Let $f : X \to X$ be a fixed-point free map of a separable metric space X, i.e., $P_1(f) = \phi$. A subset C of X is called a *color* of f if $f(C) \cap C = \phi$. Note that $f(C) \cap C = \phi$ if and only if $C \cap f^{-1}(C) = \phi$. We say that a cover $\mathcal{C}$ of X is a *coloring* of f if each element C of $\mathcal{C}$ is a color of f. The minimal cardinality $C(f)$ of closed (or open) colorings of f is called the *coloring number* of f. If $C(f) < \infty$, we say that f is *colorable*.

The following results concerning coloring numbers $C(f)$ are well-known.

Theorem 1. (Lusternik and Schnirelman [10]) *Let* $f : S^n \to S^n$ *be the antipodal map of the* n*-dimensional sphere* S^n*. Then* $C(f) = n + 2$.

Theorem 2. (Aarts, Fokkink and Vermeer [1]) *Let* $f : X \to X$ *be a fixed-point free homeomorphism of a (separable) metric space* X *with* $\dim X = n < \infty$*. Then* $C(f) \leq n + 3$.

Theorem 3. (Aarts, Fokkink and Vermeer [1]) *Let* $f : X \to X$ *be a fixed-point free map of a compact metric space* X *with* $\dim X = n < \infty$*. Then* $C(f) \leq n + 3$.

Let $f : X \to X$ be a fixed-point free map of a space X and $p \in \mathbb{N}$. A subset C of X is *eventually colored within* p of f if $\bigcap_{i=0}^{p} f^{-i}(C) = \phi$. Note that C is a color of f if and only if C is eventually colored within 1. Then we note the following simple fact which is a property of the dynamical system (X, f).

Proposition 4. *Let $f : X \to X$ be a fixed-point free map of a separable metric space X and $p \in \mathbb{N}$. Then a subset C of X is eventually colored within p of f if and only if each point $x \in C$ wanders off C within p, i.e. for each $x \in C$, $f^i(x) \notin C$ with some $1 \leq i \leq p$.*

We define the eventual coloring number $C(f, p)$ of f as follows (see [5]). A cover $\mathcal{C}$ of X is called an *eventual coloring within* p if each element C of $\mathcal{C}$ is eventually colored within p. The minimal cardinality $C(f, p)$ of all closed (or open) eventual colorings within p is called the *eventual coloring number* of f within p. Note that $C(f,1) = C(f)$. If there is some $p \in \mathbb{N}$ with $C(f,p) < \infty$, we say that f is *eventually colorable*. If there is some $p \in \mathbb{N}$ with $C(f,p) \leq k$, we say that f is *eventually k-colorable* ($k \geq 2$).

Eventual colorings are also related to zero-dimensional covers of dynamical systems. In [7], we proved that every metric n-dimensional dynamical system with zero-dimensional set of periodic points can be covered by a metric zero-dimensional dynamical system via an at most 2^n-to-one closed map.

We investigate some properties concerning Wallman compactifications of spaces and dimensions of the sets of periodic points of maps.

Theorem 5. *Let X be a finite-dimensional separable metric space and let $f : X \to X$ be a closed map with ord $(f) < \infty$. Then there exist a metric compactification γX of X and an extension $\gamma f : \gamma X \to \gamma X$ of f such that* $\dim \gamma X = \dim X$, $Cl_{\gamma X} P_i(f) = P_i(\gamma f)$ *and* $\dim P_i(f) = \dim P_i(\gamma f)$ *for each $i \in \mathbb{N}$.*

Lemma 6. (cf. the proof of [4, Theorem 1.1]) *Let X be a finite-dimensional metric space and $f : X \to X$ a closed map with ord $(f) < \infty$. Suppose that*

$$p = (1 + \dim X) \cdot (1 + ord\ (f)).$$

If Y is a closed set of X and $f|(X - Y) : (X - Y) \to X$ is fixed-point free, i.e., $f(x) \neq x$ for $x \in X - Y$, then there is an open finite coloring $\{U_i|\ 0 \leq i \leq p\}$ of $f|(X - Y)$, i.e., $\{U_i|\ 0 \leq i \leq p\}$ is an open cover of $X - Y$ such that $U_i \cap f(U_i) = \phi$ for each $0 \leq i \leq p$.

Proof of Lemma 2.2. First, we will prove the following claim (*):
Claim (*): If A is a closed set of X with $A \cap Y = \phi$ and G_i ($0 \leq i \leq p$) are any open sets of (the subspace) A such that
(1) $Cl_X(G_i) \cap f(Cl_X(G_i)) = \phi$ for each $0 \leq i \leq p$ and
(2) $ord\ \{Cl_X(G_i) |\ 0 \leq i \leq p\} \leq 1 + \dim X$,
then there are open sets H_i ($0 \leq i \leq p$) of the subspace A such that
(3) $G_i \subset H_i$ for $0 \leq i \leq p$ and $\bigcup_{0 \leq i \leq p} H_i = A$, i.e., $\{H_i |\ 0 \leq i \leq p\}$ is an open cover of A,
(4) $Cl_X(H_i) \cap f(Cl_X(H_i)) = \phi$ for each $0 \leq i \leq p$ and
(5) $ord\ \{Cl_X(H_i) |\ 0 \leq i \leq p\} \leq 1 + \dim X$.

We prove Claim (*). Since $f|A : A \to X$ is fixed-point free, for some ordinal number κ there is a locally finite open cover $\{A_\xi |\ 0 \leq \xi < \kappa\}$ of the subspace A such that $f(Cl_X(A_\xi)) \cap Cl_X(A_\xi) = \phi$. By the similar way to the proof of [4, Theorem 1.1], for each $\xi < \kappa$ we will construct an open cover $\{V_{\xi,i} |\ 0 \le i \le p\}$ of A_ξ, i.e., $\bigcup\{V_{\xi,i} |\ 0 \le i \le p\} = A_\xi$, in such a way that if $U_{\eta,i} = G_i \cup \bigcup_{\xi<\eta} V_{\xi,i}$ for $0 \le i \le p$, then for all $\eta < \kappa$

(1_η) $Cl_X(U_{\eta,i}) \cap f(Cl_X(U_{\eta,i})) = \phi$ for each $0 \le i \le p$ and

(2_η) $ord\ \{Cl_X(U_{\eta,i}) |\ 0 \le i \le p\} \le 1 + \dim X$.

We construct $\{V_{\eta,i} |\ 0 \le i \le p\}$ by the induction on $\eta < \kappa$ (see the proof of van Douwen [4, Theorem 1.1]). First, we know that the conditions (1_0) and (2_0) hold, since $\bigcup_{\xi<0} V_{\xi,i} = \phi$ and hence $U_{0,i} = G_i$. We assume that for an $\eta < \kappa$, (1_η) and (2_η) hold. Note that

$$Cl_X(U_{\eta,i}) = Cl_X(G_i) \cup \bigcup_{\xi<\eta} Cl_X(V_{\xi,i}) \subset \bigcup_{\xi<\eta} Cl_X(A_\xi) \subset A,$$

since $\{A_\xi|\ 0 \leq \xi < \kappa\}$ is locally finite. We put

$$F_i = f^{-1}(Cl_X(U_{\eta,i})) \cup f(Cl_X(U_{\eta,i})).$$

Note that

$$ord\{f^{-1}(Cl_X(U_{\eta,i}))|\ 0 \leq i \leq p\} \leq 1 + \dim X$$

and

$$ord\{f(Cl_X(U_{\eta,i}))|\ 0 \leq i \leq p\} \leq (1 + \dim X) \cdot ord(f).$$

By use of the above facts and the definition of p, we see that

$$\bigcap_{0 \leq i \leq p} F_i = \phi.$$

Note that
(a) $\{Cl_X(A_\eta) - F_i|\ 0 \leq i \leq p\}$ covers $Cl_X(A_\eta)$, and

(b) $Cl_X(U_{\eta,i}) \cap F_i = \phi$ $(0 \le i \le p)$.
Since $\dim Cl_X(A_\eta) \le \dim X$, by [4, Lemma 2.1] there is an open cover $\{V_i |\ 0 \le i \le p\}$ of $Cl_X(A_\eta)$ such that
(c) $Cl_X(A_\eta) \cap Cl_X(U_{\eta,i}) \subset V_i$,
(d) $Cl_X(V_i) \cap F_i = \phi$, and
(e) $ord\ \{Cl_X(V_i)|\ 0 \le i \le p\} \le 1 + \dim X$.

Define $V_{\eta,i} = A_\eta \cap V_i$ $(0 \le i \le p)$. Then $U_{\eta+1,i}(= U_{\eta,i} \cup V_{\eta,i})$ satisfies the conditions $(1)_{\eta+1}$, since $(1)_\eta$ and (d) hold. Also, $U_{\eta+1,i}$ satisfies the conditions $(2)_{\eta+1}$, since $(2)_\eta$, (c) and (e) hold. Consequently, we have the collection $\{U_{\eta,i}|\ \eta < \kappa, 0 \le i \le p\}$ satisfying $(1)_\eta$ and $(2)_\eta$.

Now, we put $H_i = \bigcup_{\eta<\kappa} U_{\eta,i}$ for each $0 \le i \le p$. Then H_i $(0 \le i \le p)$ are open sets of A satisfying the conditions of Claim (*).

Next, we take a sequence A_k $(k < \omega)$ of closed sets of X such that $A_k \subset Int_X(A_{k+1})$ and $\bigcup_{k<\omega} A_k = X - Y$, since Y is a closed set of the metric space X. By use of Claim (*), we have open sets $H_{0,i}$ $(0 \leq i \leq p)$ of X such that
$(i)_0$ $A_0 \subset \bigcup_{0\leq i\leq p} H_{0,i} \subset \bigcup_{0\leq i\leq p} Cl_X(H_{0,i}) \subset Int_X(A_1)$,
$(ii)_0$ $ord\ \{Cl_X(H_{0,i}) |\ 0 \leq i \leq p\} \leq 1 + \dim X$, and
$(iii)_0$ $f(Cl_X(H_{0,i})) \cap Cl_X(H_{0,i}) = \phi$ $(0 \leq i \leq p)$.

We continue this procedure recursively. By use of Claim (*), for each $k < \omega$ we obtain open sets $H_{k,i}$ $(0 \leq i \leq p)$ of X such that $H_{k,i} \subset H_{k+1,i}$ and
$(i)_k$ $A_k \subset \bigcup_{0\leq i\leq p} H_{k,i} \subset \bigcup_{0\leq i\leq p} Cl_X(H_{k,i}) \subset Int_X(A_{k+1})$,
$(ii)_k$ $ord\ \{Cl_X(H_{k,i}) |\ 0 \leq i \leq p\} \leq 1 + \dim X$, and
$(iii)_k$ $f(Cl_X(H_{k,i})) \cap Cl_X(H_{k,i}) = \phi$ $(0 \leq i \leq p)$.

Finally we put $U_i = \bigcup_{k<\omega} H_{k,i}$ $(0 \leq i \leq p)$. Then we see that U_i $(0 \leq i \leq p)$ are open sets of X satisfying the desired conditions.

Proof of Theorem 2.1. We put $\mathcal{P} = \{P_i(f)|\ i \in \mathbb{N}\}$ and enumerate $\mathcal{P}$ as $\{Q_j|\ j \in \mathbb{N}\}$ such that if $P \in \mathcal{P}$, then $|\{j \in \mathbb{N}|\ P = Q_j\}| = \infty$. i.e., each $P \in \mathcal{P}$ is listed infinitely often in $\{Q_j|\ j \in \mathbb{N}\}$. Then we have a Wallman compactification $\alpha_1 X = \omega(X, \mathcal{F}_1)$ such that $\mathcal{F}_1$ is a countable Wallman (closed) base for X and $\dim \alpha_1 X = \dim X$ and $\dim Cl_{\alpha_1 X}(Q_1) = \dim Q_1$, i.e.,

$$\omega(X, \mathcal{F}_1) = \{\mathcal{S} \subset \mathcal{F}_1|\ \mathcal{S} \text{ is an } \mathcal{F}_1\text{-ultrafilter}\}$$

(see [11, Theorem 3.5.3]). For each $F \in \mathcal{F}_1$, let

$$F^\star = \{\mathcal{S} \in \omega(X, \mathcal{F}_1)|\ F \in \mathcal{S}\}.$$

Then we know that $F^\star = Cl_{\alpha_1 X}(F)$. Suppose $Q_1 = P_{i_1}(f)$ for some $i_1 \in \mathbb{N}$. Consider the restriction $f^{i_1}|(X - Q_1) : X - Q_1 \to X$ of f^{i_1}. Note that Q_1 is an f-invariant closed subset of X, but $X - Q_1$ may not be an f-invariant subset of X. Also, note that $f^{i_1}|(X - Q_1)$ is a fixed-point free map. Since ord$(f^{i_1}) < \infty$, by lemma 2.2 $f^{i_1}|(X - Q_1)$ is colorable. By taking the closed shrinkings of the finite open colorings of $f^{i_1}|(X - Q_1)$, we have a finite closed covering $\mathcal{H}_1$ of $X - Q_1$ such that $f^{i_1}(H) \cap H = \phi$ for each $H \in \mathcal{H}_1$. Put

$$\mathcal{K}_1 = \{Q_1, Cl_X(H) |\ H \in \mathcal{H}_1\}.$$

Then $\mathcal{K}_1$ is a finite closed covering of X. Since f is a closed map of X, we can choose a countable Wallman base $\mathcal{F}_1'$ such that $\mathcal{K}_1 \cup \mathcal{F}_1 \subset \mathcal{F}_1'$, $f(\mathcal{F}_1') \subset \mathcal{F}_1'$ and $f^{-1}(\mathcal{F}_1') \subset \mathcal{F}_1'$. Put $t_1 X = \omega(X, \mathcal{F}_1')$. Then we have the unique extension $f_1 : t_1 X \to t_1 X$ of f and the extension $a_1 : t_1 X \to \alpha_1 X$ of $id_X : X \to X$.

Then we will show that $P_{i_1}(f_1) = Cl_{t_1X}(P_{i_1}(f))(= (P_{i_1}(f))^\star)$. It is obvious that $P_{i_1}(f_1) \supset Cl_{t_1X}(P_{i_1}(f)) = (P_{i_1}(f))^\star$. We show that the converse inclusion holds. Suppose, on the contrary, that there is a point $q \in P_{i_1}(f_1) - Cl_{t_1X}(P_{i_1}(f)) \subset \omega(X, \mathcal{F}'_1)$. Since $\mathcal{K}_1$ is a finite closed covering of X, there is some $Cl_X(H)$ $(H \in \mathcal{H}_1)$ with $q \in Cl_{t_1X}(Cl_X(H))(= (Cl_X(H))^\star)$. Since $P_{i_1}(f) \in \mathcal{F}'_1$ and $P_{i_1}(f) \notin q$, we can choose $F \in \mathcal{F}'_1$ such that $F \in q$ and $F \cap P_{i_1}(f) = \phi$. Then $F \cap Cl_X(H)(= E) \in \mathcal{F}'_1$, $E \in q$ and $E \subset H$. Hence $f^{i_1}(E) \cap E = \phi$. Since $E, f^{i_1}(E) \in \mathcal{F}'_1$, we see that $(f^{i_1}(E))^\star \cap E^\star = \phi$. Then we see that

$$f_1^{i_1}(q) \in f_1^{i_1}(E^\star) = f_1^{i_1}(Cl_{t_1X}(E)) \subset Cl_{t_1X}(f_1^{i_1}(E)) = Cl_{t_1X}(f^{i_1}(E)) = (f^{i_1}(E))$$

Since $q \in E^\star$ and $f_1^{i_1}(q) \in (f^{i_1}(E))^\star$, we see that $f_1^{i_1}(q) \neq q$ and hence q is not contained in $P_{i_1}(f_1)$. This is a contradiction. Hence we see that $P_{i_1}(f_1) = Cl_{t_1X}(P_{i_1}(f))$.

Also, we can choose a Wallman compactification $\alpha_2 X = \omega(X, \mathcal{F}_2)$ such that
(1) $\mathcal{F}_2$ is a countable Wallman (closed) base for X,
(2) $\dim \alpha_2 X = \dim X$, $\dim Cl_{\alpha_2 X}(Q_2) = \dim Q_2$ and
(3) $\mathcal{F}'_1 \subset \mathcal{F}_2$ (see [11, Theorem 3.5.3]).
Then we have the extension $b_1 : \alpha_2 X \to t_1 X$ of $id_X : X \to X$. This is the first step.

If we continue this procedure recursively, we obtain the following commutative diagram

$$\begin{array}{ccccccccccccc} \alpha_1 X & \leftarrow & t_1 X & \leftarrow & \alpha_2 X & \leftarrow & t_2 X & \leftarrow & \alpha_3 X & \leftarrow & t_3 X & \leftarrow \cdots\cdots & \alpha_\infty X \\ & & \downarrow f_1 & & & & \downarrow f_2 & & & & \downarrow f_3 & & \downarrow f_\infty \\ \alpha_1 X & \leftarrow & t_1 X & \leftarrow & \alpha_2 X & \leftarrow & t_2 X & \leftarrow & \alpha_3 X & \leftarrow & t_3 X & \leftarrow \cdots\cdots & \alpha_\infty X \end{array}$$

satisfying the following properties;
(4) $\alpha_j X$ and $t_j X$ are compactifications of X for each $j \in \mathbb{N}$ such that

$\dim \alpha_j X = \dim X$, $\dim Cl_{\alpha_j X}(Q_j) = \dim Q_j$,
(5) $a_j : t_j X \to \alpha_j X$ and $b_j : \alpha_{j+1} X \to t_j X$ are extensions of $id_X : X \to X$ and
(6) $f_j : t_j X \to t_j X$ is an extension of f such that $P_{i_j}(f_j) = Cl_{t_j X}(P_{i_j}(f))$, where $P_{i_j}(f) = Q_j$.

Consider the inverse limit $\alpha_\infty X$ of the inverse sequence

$$\alpha_1 X \leftarrow t_1 X \leftarrow \alpha_2 X \leftarrow t_2 X \leftarrow \alpha_3 X \leftarrow t_3 X \leftarrow \cdots$$

and the sequence $\{f_j\}_{j=1}^{\infty}$ of maps as above. Since $b_{j-1} \cdot a_j \cdot f_j | X = f = f_{j-1} \cdot b_{j-1} \cdot a_j | X$, $b_{j-1} \cdot a_j \cdot f_j = f_{j-1} \cdot b_{j-1} \cdot a_j$. Then we have the natural map $f_\infty : \alpha_\infty X \to \alpha_\infty X$ induced by the sequence $\{f_j\}_{j=1}^{\infty}$ of maps. By the constructions, we see that $\alpha_\infty X$ is a compactification of X and the map $f_\infty : \alpha_\infty X \to \alpha_\infty X$ is an extension of f. Let $i \in \mathbb{N}$. Put

$$\{j \in \mathbb{N}|\ P_i(f) = Q_j\} = \{i(k)|\ k \in \mathbb{N}\}.$$

We assume $i(1) < i(2) < \cdots$. Then the set $P_i(f_\infty)$ is the inverse limit of the inverse sequence

$$Cl_{\alpha_{i(1)}X}(P_i(f)) \ \leftarrow\ P_i(f_{i(1)}) \ \leftarrow\ \cdots\ \ Cl_{\alpha_{i(2)}X}(P_i(f)) \ \leftarrow\ P_i(f_{i(2)}) \ \leftarrow\ \cdots$$

Then we see that $\dim P_i(f) = \dim P_i(f_\infty)$ and $Cl_{\alpha_\infty X} P_i(f) = P_i(f_\infty)$ for each $i \in \mathbb{N}$. Hence if we put $\gamma X = \alpha_\infty X$ and $\gamma f = f_\infty$, then γX and γf satisfy the desired conditions.

Corollary 7. (cf. [4, Theorem 1.1]) *Let X be a finite-dimensional separable metric space and let $f : X \to X$ be a closed map with ord $(f) < \infty$. If $P(f) = \phi$, then there exist a metric compactification γX of X and an extension $\gamma f : \gamma X \to \gamma X$ of f such that* $\dim \gamma X = \dim X$ *and $P(\gamma f) = \phi$.*

In the proof of Theorem 2.1, we can take the set 2^X of all closed sets of X as the Wallman base for X. By the same arguments as in the proof of Theorem 2.1, we can prove the following corollary about the Stone-Čech compactification βX. In Corollary 2.4, if we assume $P_1(f) = \phi$, then the corollary is the theorem of van Douwen.

Corollary 8. *Let X be a finite-dimensional separable metric space and let $f : X \to X$ be a closed map with ord $(f) < \infty$. Then $Cl_{\beta X} P_i(f) = P_i(\beta f)$ for each $i \in \mathbb{N}$.*

For evaluating the eventual coloring numbers of maps, we have been defined the following indices $\psi_n(k)$ and $\tau_n(k)$.

For each $n \in \mathbb{N} \cup \{0\}$ and $k = 0, 1, 2, ..., n + 1$, we defined the index $\varphi_n(k)$ as follows. Put $\varphi_n(0) = 1$ ($k = 0$). For each $k = 1, 2, ..., n + 1$, by induction on k we defined the index $\varphi_n(k)$ by

$$\varphi_n(k) = 2\varphi_n(k-1) + [\frac{n}{n+2-k}] \cdot (\varphi_n(k-1) + 1),$$

where $[x] = \max\{m \in \mathbb{N} \cup \{0\}| \ m \leq x\}$ for $x \in [0, \infty)$. Also, for each $n \in \mathbb{N} \cup \{0\}$ and $k = 0, 1, 2, ..., n + 1$, we defined the index $\tau_n(k)$ by

$$\tau_n(k) = k(2n + 1) + 1.$$

Now we will construct new index as follows. Let $n \in \mathbb{N} \cup \{0\}$ and $0 \leq k \leq n + 1$. Put $R(n, k) = n - (n + 2 - k)[\frac{n}{n+2-k}]$, where $[x] =$

max$\{m \in \mathbb{N} \cup \{0\} | \ m \leq x\}$ for $x \in [0, \infty)$. Note that $R(n,k)$ means the remainder of n divided by $(n+2-k)$. First, we put $\psi_n(0) = 1$ $(k = 0)$. Next we consider the following two cases (i) and (ii):

$$\text{(i) } R(n,k) < n + 1 - k.$$

$$\text{(ii) } R(n,k) = n + 1 - k.$$

For each $1 \leq k \leq n + 1$, we define the index $\psi_n(k)$ by

$$\psi_n(k) = \begin{cases} k(2[\frac{n}{n+2-k}] - 1) + 2 & (\text{if } R(n,k) < n + 1 - k), \\ k(2[\frac{n}{n+2-k}] + 1) + 1 & (\text{if } R(n,k) = n + 1 - k). \end{cases}$$

Proposition 9. *For the indices $\varphi_n(k), \tau_n(k)$ and $\psi_n(k)$, the following inequalities hold.*

1. *For each $n \in \mathbb{N} \cup \{0\}$ and $0 \leq k \leq n+1$,*

$$\psi_n(k) \leq \min\{\varphi_n(k), \tau_n(k)\}.$$

2. *For each $n \geq 2$ and $2 \leq k \leq n$,*

$$\psi_n(k) < \min\{\varphi_n(k), \tau_n(k)\}.$$

3. *Moreover, for each $n \in \mathbb{N} \cup \{0\}$,*

$$\psi_n(0) = \varphi_n(0) = \tau_n(0) = 1, \;\; \psi_n(1) = \varphi_n(1) = 2, \;\; \psi_n(n+1) = \tau_n(n+1).$$

Tables of three indices $\varphi_n(k), \tau_n(k)$ and $\psi_n(k)$

$\varphi_n(k)$

nk	0	1	2	3	4	5	6
0	1	2	-	-	-	-	-
1	1	2	7	-	-	-	-
2	1	2	7	30	-	-	-
3	1	2	7	22	113	-	-
4	1	2	7	22	90	544	-
5	1	2	7	22	69	278	1951

$\tau_n(k)$

nk	0	1	2	3	4	5	6
0	1	2	-	-	-	-	-
1	1	4	7	-	-	-	-
2	1	6	11	16	-	-	-
3	1	8	15	22	29	-	-
4	1	10	19	28	37	46	-
5	1	12	23	34	45	56	67

$\psi_n(k)$

nk	0	1	2	3	4	5	6
0	1	2	-	-	-	-	-
1	1	2	7	-	-	-	-
2	1	2	4	16	-	-	-
3	1	2	4	10	29	-	-
4	1	2	4	5	14	46	-
5	1	2	4	5	13	26	67

In [5] and [6], we showed the following theorem for the case that $f : X \to X$ is a homeomorphism on a separable metric space X.

Theorem 10. ([5, 6]) *If X is a separable metric space with $\dim X = n < \infty$ and $f : X \to X$ is a fixed-point free homeomorphism with $\dim P(f) \leq 0$, then f is eventually 2-colorable. More precisely,*

$$C(f, \psi_n(k)) \leq n + 3 - k$$

for each $k = 0, 1, 2, ..., n + 1$.

For the case that $f : X \to X$ is a map of a compact metric space X, we have the following theorem.

Theorem 11. ([5, 6]) *If X is a compact metric space with $\dim X = n < \infty$ and $f : X \to X$ is any fixed-point free map with $\dim P(f) \leq 0$, then f is eventually 2-colorable. More precisely,*

$$C(f, \psi_n(k)) \leq n + 3 - k$$

for each $k = 0, 1, 2, ..., n + 1$.

In this section, we prove the following theorem which strengthens Theorem 3.1.

Theorem 12. (cf. [5, 6]) *Suppose that* X *is a separable metric space with* $\dim X = n < \infty$ *and* $f : X \to X$ *is a fixed-point free closed map such that* $\dim P(f) \leq 0$. *If ord* $(f) < \infty$, *then* f *is eventually 2-colorable. More precisely,*

$$C(f, \psi_n(k)) \leq n + 3 - k$$

for each $k = 0, 1, 2, ..., n + 1$.

Proof of Theorem 3.3. By Theorem 2.1, we have a metric compactification γX of X and an extension $\gamma f : \gamma X \to \gamma X$ of f such that $\dim \gamma X = \dim X$, $\dim P_1(\gamma f) = -1$ and $\dim P(\gamma f) \leq 0$. Since γX is a compact metric space, by Theorem 3.2 we see that $\gamma f : \gamma X \to \gamma X$ is eventually 2-colorable and moreover we see that $C(\gamma f, \psi_n(k)) \leq n + 3 - k$ for each $k = 0, 1, 2, ..., n + 1$. Thus f satisfies the desired properties;

$$C(f, \psi_n(k)) \leq n + 3 - k$$

for each $k = 0, 1, 2, ..., n + 1$.

Next, we will consider the case that X is a locally compact separable metric space. In [2, Theorem 3.4 and Proposition 4.3], Buzyakova and Chigogidze proved the following.

Theorem 13. ([2]) *If X is a locally compact (paracompact) space with* $\dim X < \infty$ *and $f : X \to X$ is any fixed-point free map, then f is colorable and moreover, $Cl_{\beta X} P_i(f) = P_i(\beta f)$ for each $i \in \mathbb{N}$.*

By use of the above theorem, we obtain the following result which strengthens Theorem 3.2. For the proof, we need the theory of dimensions for non-metrizable spaces (e.g., see [3]).

Theorem 14. (cf. [5, 6]) *Let X be a locally compact separable metric space with* $\dim X = n < \infty$. *If $f : X \to X$ is any fixed-point free map with* $\dim P(f) \leq 0$, *then f is eventually 2-colorable. More precisely,*

$$C(f, \psi_n(k)) \leq n + 3 - k$$

for each $k = 0, 1, 2, ..., n + 1$.

Proof. By Theorem 3.4, we see that $\dim P_i(\beta f) \leq 0$ for each $i \in \mathbb{N}$, and hence $\dim P(\beta f) \leq 0$. Note that $\beta f : \beta X \to \beta X$ is a fixed-point free map of a compact Hausdorff space βX satisfying $\dim P(\beta f) \leq 0$. By some generalized arguments, we see that Theorem 3.2 is also true for the case that X is a compact Hausdorff space (see the proofs of [5, Theorem 3.1 and Corollary 3.2]). Consequently, we see that f is eventually 2-colorable.

Corollary 15. *Suppose that one of the following two conditions (a) and (b) is satisfied: (a)* X *is an* n*-dimensional separable metric space and* $f : X \to X$ *is a fixed-point free closed map such that* $\dim P(f) \leq 0$ *and ord* $(f) < \infty$*. (b)* X *is an* n*-dimensional locally compact separable metric space and* $f : X \to X$ *is any fixed-point free map with* $\dim P(f) \leq 0$*. Then*

$$C(f, \psi_n(n+1)) \leq 2.$$

In particular, the following statements hold.
(0) If $\dim X = 0$*, then* $C(f, 2) = 2$*.*
(1) If $\dim X = 1$*, then* $C(f, 7) = 2$*.*
(2) If $\dim X = 2$*, then* $C(f, 16) = 2$*.*
(3) If $\dim X = 3$*, then* $C(f, 29) = 2$*.*
(4) If $\dim X = 4$*, then* $C(f, 46) = 2$*.*
(5) If $\dim X = 5$*, then* $C(f, 67) = 2$*.*

Finally, we have the following problems.

Problem 16. (cf. [2, Question 4.6]) *Let X be a σ-compact space with* $\dim X < \infty$*. Is every fixed-point free map $f : X \to X$ with* $\dim P(f) \leq 0$ *eventually 2-colorable?*

Problem 17. *Let $f : X \to X$ be a map of a locally compact separable metric space X with* $\dim X < \infty$*. Is it true that there exist a metric compactification γX of X and an extension $\gamma f : \gamma X \to \gamma X$ of f such that* $\dim \gamma X = \dim X$, $Cl_{\gamma X} P_i(f) = P_i(\gamma f)$ *and* $\dim P_i(f) = \dim P_i(\gamma f)$ *for each $i \in \mathbb{N}$?*

1. J. M. Aarts, R. J. Fokkink and H. Vermeer, Variations on a theorem of Lusternik and Schnirelmann, *Topology*, 35 (1996), 1051-1056.

2. R. Z. Buzyakova and A. Chigogidze, Fixed-point free maps of Euclidean spaces, *Fund. Math*. 212 (2011), 1-16.

3. R. Engelking, Theory of Dimensions Finite and Infinite, *Heldermann Verlag, Lemgo,* 1995.

4. E. K. van Douwen, βX and fixed-point free map, *Topology Appl.* 51 (1993), 191-195.

5. Y. Ikegami, H. Kato and A. Ueda, Eventual colorings of homeomorphisms, *J. Math. Soc. Japan*, to appear.

6. Y. Ikegami, H. Kato and A. Ueda, On eventual coloring numbers, *Topology Proceedings*, to appear.

7. Y. Ikegami, H. Kato and A. Ueda, Dynamical systems of finite-dimensional metric spaces and zero-dimensional covers, preprint.

8. M. Katětov, A theorem on mappings, *Comm. Math. Univ. Carolin.* 8 (1967), 431-433.

9. A. Krawczyk and J. Steprāns, Continuous colourings of closed

graphs, *Topology Appl.* 51,(1993), 13-26.

10. L. Lusternik and L. Schnirelman, Topological methods in variational calculus (Russian), *Moscow*, 1930.

11. J. van Mill, The Infinite-Dimensional Topology of Function Spaces, *North-Holland publishing Co., Amsterdam,* 2001.