Reflection Theorems
in
Topology

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Reflection and non-reflection

Theorem 1. (A. Dow, 1988)

For any <u>*countably compact*</u> *Hausdorff space X if all subspaces of X of cardinality $\leq \aleph_1$ are metrizable then X is also metrizable.*

Theorem 2. (A. Hajnal and I. Juhász, 1976)

For any uncountable cardinal κ there is a non-metrizable space X s.t. all subspaces Y of X of cardinality $< \kappa$ are metrizable.

Sketch of the Proof

Theorem 3. (S.F., I.Juhász, L.Soukup, Z.Szentmiklóssy and T.Usuba 2011, S.F., Soukup, H.Sakai and Usuba, 201?) *The following assertion is equivalent with the* <u>*Fodor-type reflection principle*</u> *(FRP) over ZFC:*

For any <u>*locally countably compact Hausdorff*</u> *space X if all subspaces of X of cardinality $\leq \aleph_1$ are metrizable then X is also metrizable.*

Reflection Theorems equivalent to FRP

▶ The following assertions among many others are also kown to be equivalent to FRP over ZFC:

> **(S.F., 2008, S.F., L.Soukup, H.Sakai and T.Usuba, 201?)** *For every T_1-space X with point countable base, if all subspaces of X of cardinality $\leq \aleph_1$ are* <u>left-separated</u> *then X itself is also left-separated.*

> **(S.F., I.Juhász, L.Soukup, Z.Szentmiklóssy and T.Usuba 2011, S.F., Soukup, H.Sakai and Usuba, 201?)** *For every locally separable countably tight topological space X, if all subspaces of X of cardinality $\leq \aleph_1$ are* <u>meta-Lindelöf</u>, *then X itself is also meta-Lindelöf.*

> **(S.F., Soukup, H.Sakai and Usuba, 201?)** *For every countably tight topological space X of local density $\leq \aleph_1$, if all subspace of cardinality $\leq \aleph_1$ are* <u>collectionwise Hausdorff</u>, *then X is collectionwise Hausdorff.*

Proof of the equivalences

▶ The "⇒" direction of the equivalence proofs are quite involved (we skip it here).

▶ Most of the implications "¬FRP ⇒ ···" can be proved by the following construction of topological spaces:

> **Fact.** *If ¬FRP holds then there is a regular cardinal λ with $ADS^-(\lambda)$: there are stationary $E^* \subseteq E_\omega^\lambda$ and a ladder system $g^* : E^* \to [\lambda]^{\aleph_0}$ s.t. $g^* \restriction \alpha$ is essentially disjoint for all $\alpha < \lambda$.*

> **Fact.** If $\neg$FRP holds then there is a regular cardinal λ with $ADS^-(\lambda)$: there are stationary $E^* \subseteq E_\omega^\lambda$ and a ladder system $g^* : E^* \to [\lambda]^{\aleph_0}$ s.t. $g^* \restriction \alpha$ is essentially disjoint for all $\alpha < \lambda$.

▶ Let λ, E^*, g^* be as above. We may assume that $g^*(\alpha) \cap E^* = \emptyset$ for all $\alpha \in E^*$.

▶ Let $X = E^* \cup \bigcup_{\alpha \in E^*} g^*(\alpha)$ and $\mathcal{O}$ be the topology on X generated from
$$\mathcal{B} = \{\{\alpha\} \,:\, \alpha \in \bigcup_{\alpha \in E^*} g^*(\alpha)\}$$
$$\cup \{g^*(\alpha) \cup \sup\{\alpha\} \setminus x \,:\, \alpha \in E^*, x \in [g^*(\alpha)]^{<\aleph_0}\}$$

▶ Any subspace Y of X of cardinality $< \lambda$ is metrizable: Since $g^*(\alpha)$, $\alpha \in E^* \cap Y$ are essentially disjoint, Y can be partitioned into disjoint metrizable open subspaces.

▶ X itself is not metrizable since it is not meta-Lindelöf: Consider the open $\mathcal{B}$. Fodor's Lemma imples that there is no point countable open refinement.

▶ Recall the reflection theorems on metrizability on the first slide.

▷ What is about the metrizability of spaces of countable character?

▷ Restriction to the spaces of countable tightness already makes the reflection number of metrizability consistently $< \infty$:

> **Theorem 4.** Suppose that κ is a/the ω_1-strongly compact cardinal. For any countably tight X, if all subspaces of X of cardinality $< \kappa$ are metrizable, then X itself is also metrizable.

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Proof.

▶ For X as above, let $A = X \,\dot\cup\, [X]^{\aleph_0} \,\dot\cup\, \mathbb{R}$.

▶ Let $E, R \subseteq A$ be defined by

$x \mathrel{E} y :\Leftrightarrow x \in X$, $y \in [X]^{\aleph_0}$ and $x \in y$; and

$x \mathrel{R} y :\Leftrightarrow x \in X$, $y \in [X]^{\aleph_0}$ and $x \in \overline{y}$.

Note that R decides the topology of X. Let $\leq$ be the canonical less than or equal to relation on $\mathbb{R}$.

▶ Let T be the $\mathcal{L}_{\omega_1,\omega}$ theory in $L = \{E, R, d(.,.), \leq, c_a\}_{a \in A}$ consisting of quantifier free diagram of $\langle A, E, R \rangle$ and the assertion "m is a metric generating the topology introduced by R".

▶ Then T is $< \kappa$-satisfiable. Since κ is ω_1-strongly compact, T is satisfiable. A metric of X can be constructed from a model of T. □

▶ We can consider the following cardinal numbers (they can be ∞):

$\mathfrak{Refl} = \min\{\kappa :$ for all <u>locally compact</u> X,
if all supspace of X of card. $< \kappa$ are metrizable,
then X is also metrizable $\}$;

$\mathfrak{Refl}^* = \min\{\kappa :$ for all <u>countably tight</u> X,
if all supspace of X of card. $< \kappa$ are metrizable,
then X is also metrizable $\}$;

▶ $\aleph_1 \leq \mathfrak{Refl} = \mathfrak{Refl}_{\mathsf{FRP}} \leq \mathfrak{Refl}^* \leq \omega_1$-strongly compact cardinal

<u>Open questions</u>:

▷ What is the possible value of $\mathfrak{Refl}^*$? Can it be $\aleph_1$?

▷ Can $\mathfrak{Refl}$ or $\mathfrak{Refl}^*$ be ω_1-strongly compact?

▶ $\aleph_1 \leq \mathfrak{Refl} \leq \mathfrak{Refl}_{Rado} \leq \mathfrak{Refl}_{Galvin}$
$\leq \mathfrak{Refl}_{chr} \leq \omega_1$-strongly compact cardinal

▷ For the definition of these cardinals see e.g.
http://kurt.scitec.kobe-u.ac.jp/~fuchino/slides/wien12-06-22-pf.pdf

▶ $\mathfrak{Refl}_{Rado} = \aleph_1$ is Rado Conjecture — proved to be consistent
(modulo some large cardinals) by S. Todorčević.

▶ $\beth_\omega \leq Refl_{chr}$ (Erdő and Hajnal, 1968)

▷ $\mathfrak{Refl}_{Galvin} = \aleph_1$ is Galvin Conjecture — status: open.

▷ Is there any relationship between $\mathfrak{Refl}_{Rado}$, $\mathfrak{Refl}_{Galvin}$, $\mathfrak{Refl}^*$?

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Sketch of the proof of Theorem 2

Theorem 2. (A. Hajnal and I. Juhász, 1976)
For any uncountable cardinal κ there is a non-metrizable space X s.t. all subspaces Y of X of cardinality $< \kappa$ are metrizable.

Proof. For every cardinal $\kappa' \geq \kappa$ of uncountable cofinality, $(\kappa' + 1, \mathcal{O})$ with $\mathcal{O} = \mathcal{P}(\kappa') \cup \{(\kappa' \setminus x) \cup \{\kappa'\} : x \in [\kappa']^{<\kappa'}\}$ is such a space:

▶ Any subspace of size $< \kappa'$ is discrete and hence metrizable.
▶ $\kappa' + 1$ itself is not metrizable since κ' has character $\geq \kappa' > \aleph_0$. □

Fodor-type reflection principle (FRP)

(FRP):

For any regular uncountable λ and any stationary $S \subseteq E^{\lambda}_{\omega} = \{\alpha < \lambda : \operatorname{cf}(\alpha) = \omega\}$ and any mapping $g : S \to [\lambda]^{\leq \aleph_0}$ there is $I \in [\lambda]^{\aleph_1}$ (a reflection point of g) s.t.

(1) $\operatorname{cf}(I) = \omega_1$;

(2) $g(\alpha) \subseteq I$ for all $\alpha \in I \cap S$;

(3) for any regressive $f : S \cap I \to \lambda$ s.t. $f(\alpha) \in g(\alpha)$ for all $\alpha \in S \cap I$, there is $\xi^* < \lambda$ s.t. $f^{-1}\,''\{\xi^*\}$ is stationary in $\sup(I)$.

Left-separated topological spaces

A topological space X is **left-separated** if there is a well-ordering $<$ of X s.t. all initial segments of X w.r.t. $<$ are closed subsets of X.

$$\cdots \,](\, \cdots$$

meta-Lidelöf, and collectionwise Hausdorff spaces

A topological space X is said to be **meta-Lindelöf** if every open cover of X has an open refinement which is point countable.

T_1 space is **collectionwise Hausdorff** if, for any closed and discrete $D \subseteq X$, there is a family $\mathcal{U}$ of pairwise disjoint open sets which simultaneously separates D, that is, for all $d \in D$, there is $U \in \mathcal{U}$ with $D \cap U = \{d\}$.

ADS$^-(\lambda)$

For $X \subseteq \lambda$, $g : X \to [\lambda]^{\aleph_0}$ is a **ladder system** if $otp(g(\alpha)) = \omega$ and $g(\alpha)$ is a cofinal subset of α for all $\alpha \in X$.

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$g : X \to \mathcal{P}(Y)$ is **essentially disjoint** if there is $h : X \to [Y]^{<\aleph_0}$ s.t. $g(x) \setminus h(x)$, $x \in X$ are pairwise disjoint.

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ω_1-strongly compact cardinal

A cardinal κ is ω_1-**strongly compact** if it is the smallest κ with the property that, for any $\mathcal{L}_{\omega_1,\omega}$ theory T if all subtheories of T of cardinality $< \kappa$ are satisfiable (i.e. T is $< \kappa$-satisfiable) then T itself is satisfiable.

Countable tightness

A topological space X is **countably tight** if, for any $x \in X$ and $Y \subseteq X$ with $x \in \overline{Y}$, there is $Y' \in [Y]^{\leq \aleph_0}$ s.t. $x \in \overline{Y'}$.