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平成１４年科学研究費基盤研究（A）
位相幾何学の総合的研究（課題番号 12304003）
代表者：　東京大学　坪井　俊

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COVERING PROPERTIES OF INVERSE LIMITS, II

Keiko Chiba and Yukinobu Yajima

Throughout this report, all spaces are topological spaces without any separation axiom, and all maps are continuous. For an inverse system $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ and its limit X, let Λ be a directed set with an order $<$ and its cardinality λ, where $\lambda \geq \omega$, and let π_α be the projection from X into X_α for each $\alpha \in \Lambda$.

1. Known results and Qusetions

The following result of Bešlagić is a motivation of the study for the covering properties of inverse limits.

Theorem 1.1 [Be] (see [C1]). *Let $X = \prod_{\alpha\in\Gamma} X_\alpha$ be a product space such that $\prod_{\alpha\in F} X_\alpha$ is normal for each finite $F \subset \Gamma$. Then X is normal if and only if it is λ-paracompact, where λ is the cardinality of Γ.*

However, the "if" part of Theorem 1.1 had been already extended by Aoki:

Theorem 1.2 [A]. *Let $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ be an inverse system and X its inverse limit with each projection π_α being a pseudo-open map. Suppose that X is λ-paracompact. If each X_α is normal, then so is X.*

These results lead us to consider the following general statement:

Statement (∗). *Let $\mathcal{P}$ be a topological property. Let $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ be an inverse system and X its inverse limit with each projection π_α being a pseudo-open map. Suppose that X is λ-paracompact. If each X_α is $\mathcal{P}$, then so is X.*

Aoki and Chiba proved many results for $\mathcal{P}$ being several covering properties and some other separation properties in the Statement (∗) as follows.

Theorem 1.3 [A, C2, C4]. *Let $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ be an inverse system and X its inverse limit with each projection π_α being a pseudo-open map. Suppose that X is λ-paracompact. If each X_α satisfies one of the following properties, then X has the corresponding property.*

(1) *Paracompactness.*
(2) *Collectionwise normality.*
(3) *Subparacompactness.*
(4) *Metacompactness.*
(5) *Submetacompactness (= θ-refinability).*
(6) *Subnormality.*

Moreover, we can consider another general statement:

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Statement (∗∗). *Let $\mathcal{P}$ be a topological property. Let $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ be an inverse system and X its inverse limit. Suppose that X is hereditarily λ-paracompact. If each X_α is hereditarily $\mathcal{P}$, then so is X.*

Chiba also proved a similar result to Theorem 1.3 for the Statement (∗∗) as follows.

Theorem 1.4 [C2, C4]**.** *Let $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ be an inverse system and X its inverse limit. Suppose that X is hereditarily λ-paracompact. If each X_α satisfies one of the following properties, then X has the corresponding property.*

(0) *Hereditary normality.*
(1) *Hereditary paracompactness.*
(2) *Hereditary collectionwise normality.*
(3) *Hereditary subparacompactness.*
(4) *Hereditary metacompactness.*
(5) *Hereditary submetacompactness (= hereditary θ-refinability).*
(6) *Hereditary subnormality.*

The purpose of this study is to prove that Theorems 1.3 and 1.4 hold for all main covering properties and all main separation properties. From this point of view, it is natural to raise the following three questions:

Question 1 [C4]**.** (i) Does Theorem 1.3 hold for $\delta\theta$-refinability?
(ii) Does Theorem 1.4 hold for hereditary $\delta\theta$-refinability?

Question 2 [C4]**.** (i) Does Theorem 1.3 hold for collectionwise δ-normality?
(ii) Does Theorem 1.4 hold for hereditary collectionwise δ-normality?

Question 3 [C4]**.** (i) Does Theorem 1.3 hold for collectionwise subnormality?
(ii) Does Theorem 1.4 hold for hereditary collectionwise subnormality?

As a partial answer of Question 1 (i), Chiba obtained

Theorem 1.5 [C3, C4]**.** *Let $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ be an inverse system and X its inverse limit with each projection π_α being a pseudo-open map. Suppose that X is λ-paracompact.*

(1) *If each X_α is normal and $\delta\theta$-refinable, then X is $\delta\theta$-refinable.*
(2) *If each X_α is $\delta\theta$-refinable and Λ is countable, then X is $\delta\theta$-refinable.*

Remark. It should be noted that a certain assumption of X such as λ-paracompactness seems to be always necessary to consider the covering properties of inverse limits. In fact, the product ω^{ω_1} of uncountably many copies of ω (= the countable infinite discrete space) is the limit of an inverse system of discrete spaces with each projection being open. However, ω^{ω_1} is not countably paracompact and not subnormal and it is not even weakly $\delta\theta$-refinable (see [CGP, 11.4]).

2. $\delta\theta$-refinability and weak θ-refinability

A map f from X onto Y is *pseudo-open* if $y \in \operatorname{Int} f(U)$ holds for each $y \in Y$ and each open set U in X with $f^{-1}(y) \subset U$. Note that both open and onto maps and closed and onto maps are pseudo-open.

A space X is *λ-paracompact* if every open cover of X with cardinality $\leq \lambda$ has a locally finite open refinement. A cover $\mathcal{A}$ of a space X is *directed* if for any $A_0, A_1 \in \mathcal{A}$, there is $A_2 \in \mathcal{A}$ with $A_0 \cup A_1 \subset A_2$.

Lemma 2.1 [M]**.** *A space X is λ-paracompact if and only if for every directed open cover $\mathcal{U}$ of X with cardinality $\leq \lambda$, there is a locally finite open cover $\mathcal{V}$ of X such that $\{\overline{V}\colon V \in \mathcal{V}\}$ refines $\mathcal{U}$.*

Recall that a space X is *weakly θ-refinable* (respectively, *weakly $\delta\theta$-refinable*) if for every open cover $\mathcal{U}$ of X, there is an open refinement $\bigcup_{n\in\omega} \mathcal{V}_n$ of $\mathcal{U}$ such that for each $x \in X$ one can find $n_x \in \omega$ with $\mathrm{ord}(x, \mathcal{V}_{n_x}) = 1$ (respectively, $1 \leq \mathrm{ord}(x, \mathcal{V}_{n_x}) \leq \omega$).

Before we consider the Qustion 1 (i), we should also consider the following similar question:

Question 1′. Does Theorem 1.3 hold for weak θ-refinability or weak $\delta\theta$-refinability? $\blacksquare$

First, we can obtain an affirmative answer to this question as follows.

Theorem 2.2. *Let $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ be an inverse system and X its inverse limit with each projection π_α being a pseudo-open map. Suppose that X is λ-paracompact. If each X_α is weakly θ-refinable (weakly $\delta\theta$-refinable), then so is X.*

Next, we proceed to consider the Question 1 (i). For that, the following concept plays an important role.

A space X is *λ-subparacompact* if every open cover of X with cardinality $\leq \lambda$ has a σ-locally finite closed refinement. Let X be a space and $\mathcal{V}$ a collection of subsets in X. For each $x \in X$, we denote by $\mathrm{ord}(x, \mathcal{V})$ the cardinality of $\{V \in \mathcal{V}\colon x \in V\}$.

Lemma 2.3 [B1, B2]**.** *For a space X, the following are equivalent.*

(a) *X is λ-subparacompact.*
(b) *Every open cover of X with cardinality $\leq \lambda$ has a σ-discrete closed refinement.*
(c) *Every open cover of X with cardinality $\leq \lambda$ has a σ-closure-preserving closed refinement.*
(d) *For every open cover $\mathcal{U}$ of X with cardinality $\leq \lambda$, there is a sequence $\{\mathcal{V}_n\}$ of open refinements of $\mathcal{U}$ such that for each $x \in X$ one can find $n_x \in \omega$ with* $\mathrm{ord}(x, \mathcal{V}_{n_x}) = 1$.

Remark. As is well-known, paracompactness implies subparacompactness. However, for each $\lambda \geq \omega$, λ-paracompactness does not imply λ-subparacompactness. In fact, let $X_\lambda = \lambda^+ \times (\lambda^+ + 1)$. Since X_λ is the product of a λ-paracompact space and a compact space, it is λ-paracompact. However, X_λ is not subnormal (because, by the pressing down lemma, $\{(\alpha, \alpha) \in X_\lambda\colon \alpha \in \lambda^+\}$ and $\lambda^+ \times \{\lambda^+\}$ cannot be separated by disjoint G_δ-sets). Since every ω-subparacompact space is subnormal, X_λ is not ω-subparacompact, hence not λ-subparacompact.

A space X is *subnormal* if for any disjoint closed sets A and B in X, there are disjoint G_δ-sets G and H such that $A \subset B$ and $B \subset H$. Note that X is subnormal if and only if every finite (or binary) open cover of X has a countable closed refinement.

Lemma 2.4. *Every λ-paracompact and subnormal space is λ-subparacompact.*

A space X is *$\delta\theta$-refinable (= submetaLindelöf)* if for every open cover $\mathcal{U}$ of X, there is a sequence $\{\mathcal{V}_n\}$ of open refinements of $\mathcal{U}$ such that for each $x \in X$ one can find $n_x \in \omega$ with $\mathrm{ord}(x, \mathcal{V}_{n_x}) \leq \omega$.

Using above lemmas, we can obtain a partial answer to the Question 1 (i). This is also a generalization of [C4, Theorem 1 (i)].

Theorem 2.5. *Let $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ be an inverse system and X its inverse limit with each projection π_α being a pseudo-open map. Suppose that X is λ-paracompact and subnormal. If each X_α is $\delta\theta$-refinable, then so is X.*

Theorems 1.3 (6) and 2.6 immediately yield

Corollary 2.6. *Let $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ be an inverse system and X its inverse limit with each projection π_α being a pseudo-open map. Suppose that X is λ-paracompact. If each X_α is $\delta\theta$-refinable and subnormal, then so is X.*

3. Collectionwise δ-normality and collectionwise subnormality

A space X is *collectionwise δ-normal* if for every discrete collection $\{C_\xi : \xi \in \Xi\}$ of subsets in X, there is a disjoint collection $\{U_\xi : \xi \in \Xi\}$ of G_δ-sets in X such that $C_\xi \subset U_\xi$ for each $\xi \in \Xi$. It is clear that every collectionwise δ-normal space is subnormal.

The following is an affirmative answer to the Question 2 (i) (= [C4, Question 3]).

Theorem 3.1. *Let $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ be an inverse system and X its inverse limit with each projection π_α being a pseudo-open map. Suppose that X is λ-paracompact. If each X_α is collectionwise δ-normal, then so is X.*

A space X is *collectionwise subnormal (= discretely subexpandable)* if for every discrete collection $\{C_\xi : \xi \in \Xi\}$ of subsets in X, there is a sequence $\mathcal{U}_n = \{U_{\xi,n} : \xi \in \Xi\}, n \in \omega$, of collections of open sets in X such that $C_\xi \subset U_{\xi,n}$ for each $\xi \in \Xi$ and each $n \in \omega$, and for each $x \in X$, one can find $n_x \in \omega$ with $\mathrm{ord}(x, \mathcal{U}_n) \leq 1$.

Note that every subparacompact space is collectionwise subnormal and that every collectionwise subnormal space is collectionwise δ-normal.

The following is an affirmative answer to Question 3 (i) (= [C4, Question 2]).

Theorem 3.2. *Let $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ be an inverse system and X its inverse limit with each projection π_α being a pseudo-open map. Suppose that X is λ-paracompact. If each X_α is collectionwise subnormal, then so is X.*

Remark. Recall that a space X is *finitely subparacompact* if every finite open cover of X has a σ-discrete closed refinement, and that a space X is *boundedly subexpandable* if X is collectionwise subnormal (= discretely subexpandable) and finitely subparacompact (see [K]). However, note that finite subparacompactness is equivalent to subnormality, and that collectionwise subnormality implies subnormality. Hence collectionwise subnormality is exactly equivalent to bounded subexpandability. So [C4, Theorem 1 (vii)] would be an affirmative answer to our Question 3 (= [C4, Question 2]) if the proof would be correct. However, there is a gap in the proof of [C4, Theorem 1 (vii)] (more precisely, the part of "Proof of (1)" is not correct). Consequently, we can consider our proof of Theorem 3.2, which is omitted here, as a correct one of [C4, Theorem 1 (vii)].

4. Hereditarily subnormality and related properties

Recall that a space X is *hereditarily subnormal* if every subspace of X is subnormal. Note that X is hereditarily subnormal if and only if every open subspace of X is subnormal.

The former part of the following was actually stated in [C4] without proof.

Proposition 4.1 [C4]**.** *Let $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ be an inverse system and X its inverse limit. Let G be an open set of X. Suppose that G is either λ-paracompact or λ-subparacompact. If each X_α is hereditarily subnormal, then G is subnormal.*

Recall that a space X is *hereditarily $\delta\theta$-refinable* if every (open) subspace of X is $\delta\theta$-refinable.

The following is a partial answer to Question 1 (ii).

Theorem 4.2. *Let $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ be an inverse system and X its inverse limit. Let G be λ-subparacompact open subspace of X. If each X_α is hereditarily $\delta\theta$-refinable, then G is $\delta\theta$-refinable.*

We also obtain a generalization of [C3, Theorem 2] as follows.

Proposition 4.3. *Let $\{X_n, \pi^n_k, \omega\}$ be an inverse sequence and X its inverse limit. Let G be a countably metacompact open subspace of X. If each X_n is hereditarily $\delta\theta$-refinable, then G is $\delta\theta$-refinable.*

A space X is *hereditarily collectionwise δ-normal* if every subspace of X is collectionwise δ-normal.

Theorem 4.4. *Let $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ be an inverse system and X its inverse limit. Let G be a λ-subparacompact open subspace of X. If each X_α is hereditarily collectionwise δ-normal, then G is collectionwise δ-normal.*

Note that a space X is hereditarily collectionwise δ-normal if and only if every open subspace of X is collectionwise δ-normal. Thus Lemma 2.4, Proposition 4.1 and Theorem 4.4 immediately yield an affirmative answer to Question 2 (ii) (= [C4, Question 6]):

Corollary 4.5. *Let $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ be an inverse system and X its inverse limit. Suppose that X is hereditarily λ-paracompact. If each X_α is hereditarily collectionwise δ-normal, then so is X.*

Recall that a space X is *hereditarily collectionwise subnormal* if every subspace of X is collectionwise subnormal.

The following gives an affirmative answer to Question 3 (ii).

Theorem 4.6. *Let $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ be an inverse system and X its inverse limit. Let G be a λ-subparacompact open subspace of X. If each X_α is hereditarily collectionwise subnormal, then G is collectionwise subnormal.*

Since a space X is hereditarily collectionwise subnormal if and only if every open subspace of X is collectionwise subnormal, Lemma 2.4, Proposition 4.1 and Theorem 4.6 immediately yield an affirmative answer to Question 3 (ii) (= [C4, Question 5]):

Corollary 4.7. *Let $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ be an inverse system and X its inverse limit. Suppose that X is hereditarily λ-paracompact. If each X_α is hereditarily collectionwise subnormal, then so is X.*

5. Hereditarily subparacompactness and hereditarily submetacompactness

We begin with the hereditary subparacompact case:

Proposition 5.1. *Let $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ be an inverse system and X its inverse limit. Let G be a λ-subparacompact open subspace of X. If each X_α is hereditarily subparacompact, then G is subparacompact.*

For the hereditary submetacompact case, we need the following lemma.

Lemma 5.2 [GY]**.** *There is a filter $\mathcal{F}$ on ω satisfying: For every submetacompact space X and every open cover $\mathcal{U}$ of X, there is a sequence $\{\mathcal{V}_n\}$ of open refinements of $\mathcal{U}$ such that for each $x \in X$, $\{n \in \omega : \operatorname{ord}(x, \mathcal{V}_n) < \omega\} \in \mathcal{F}$.*

A space X is λ-*submetacompact* if for every open cover $\mathcal{U}$ of X with cardinality $\leq \lambda$, there is a sequence $\{\mathcal{V}_n\}$ of open refinements of $\mathcal{U}$ such that for each $x \in X$ one can find $n_x \in \omega$ with $\operatorname{ord}(x, \mathcal{V}_{n_x}) < \omega$. Clearly, λ-subparacompactness implies λ-submetacompactness.

Theorem 5.3. *Let $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ be an inverse system and X its inverse limit. Let G be a λ-submetacompact open subspace of X. If each X_α is hereditarily submetacompact, then G is submetacompact.*

A space X is λ-*metacompact* if every open cover of X with cardinality $\leq \lambda$ has a point-finite open refinement. Clearly, λ-metacompactness implies λ-submetacompactness.

Proposition 5.4. *Let $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ be an inverse system and X its inverse limit. Let G be a λ-metacompact open subspace of X. If each X_α is hereditarily metacompact, then G is metacompact.*

A space X is λ-*weakly θ-refinable* if for every open cover $\mathcal{U}$ of X with cardinality $\leq \lambda$, there is an open refinement $\bigcup_{n\in\omega} \mathcal{V}_n$ of $\mathcal{U}$ such that for each $x \in X$ one can find $n_x \in \omega$ with $\operatorname{ord}(x, \mathcal{V}_{n_x}) = 1$. Similarly, we have

Proposition 5.5. *Let $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ be an inverse system and X its inverse limit. Let G be a λ-weakly θ-refinable open subspace of X. If each X_α is hereditarily weakly θ-refinable, then G is weakly θ-refinable.*

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(K. Chiba) Department of Mathematics, Faculty of Science, Shizuoka University, Shizuoka, 422-8529, Japan
E-mail address: smktiba@ipc.shizuoka.ac.jp

(Y. Yajima) Department of Mathematics, Kanagawa University, Yokohama 221-8686, Japan
E-mail address: yuki@cc.kanagawa-u.ac.jp

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TOPOLOGICAL SEQUENCE ENTROPY OF MONOTONE MAPS ON ONE-DIMENSIONAL CONTINUA

NAOTSUGU CHINEN

Abstract. Let X be either a dendrite or a graph and f a monotone map from X to itself. The main result is that every topological sequence entropy of f respect to every sequence S is zero. This implies that the topological entropy of f is equal to zero.

1. Introduction.

Let f be a continuous map from a continuum X to itself. We denote the n-fold composition f^n of f with itself by $f \circ \cdots \circ f$ and f^0 the identity map. Let us denote $f^{-i}(Y)$ the ith inverse image of an arbitrary set $Y \subset X$.

T. N. T. Goodman introduced in [G] the notion of topological sequence entropy as an extension of the concept to topological entropy. Let f be a continuous map from a compact metric space X to itself and $\mathbf{A}, \mathbf{B}$ finite open covers of X. Denote $\{f^{-m}(A) | A \in \mathbf{A}\}$ by $f^{-m}(\mathbf{A})$ for each positive integer m, $\mathbf{A} \vee \mathbf{B} = \{A \cap B | A \in \mathbf{A}, B \in \mathbf{B}\}$ and $N(\mathbf{A})$ denotes the minimal possible cardinality of a subcover chosen from $\mathbf{A}$. Let $S = \{s_i | i = 1, 2, \dots\}$ be an increasing unbounded sequence of positive integers. We define the *topological sequence entropy of f relative to a finite open cover $\mathbf{A}$ of X (respect to the sequence S)* as

$$h_S(f, \mathbf{A}) = \limsup_{n \to \infty} \frac{1}{n} \log N(\bigvee_{i=1}^{n-1} f^{-s_i}(\mathbf{A})).$$

And we define the *topological sequence entropy of f (respect to the sequence S)* as

$$h_S(f) = \sup\{h_S(f, \mathbf{A}) | \mathbf{A} \text{ is a finite open cover of } X\}.$$

If $s_i = i$ for each i, then $h_S(f)$ is equal to the standard topological entropy $h(f)$ of f introduced by Adler, Konheim and McAndrew in [AKM]. And set

$$h_\infty(f) = \sup_S h_S(f).$$

If X is a compact interval or the circle, in [FS] and [H] it was proved that f is chaotic in the sense of Li and Yoke if and only if $h_\infty(f) > 0$. If $f : [0,1] \to [0,1]$ is piecewise monotonic, Cánovas proved in [C1] that f is chaotic of type 2^∞ if and only if $h_\infty(f) = \log 2$, and that f is chaotic of type greater than 2^∞ if and only if $h_\infty(f) = \infty$.

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The *mesh* of an open cover $\mathbf{A}$ of X is the supremum of the diameter of the elements of $\mathbf{A}$, denoted by mesh$\mathbf{A}$. For an open cover $\mathbf{A}$ of X, we set $\mathrm{Bd}(\mathbf{A}) = \bigcup\{\mathrm{Bd}(A)|A \in \mathbf{A}\}$, where $\mathrm{Bd}(Y)$ denotes the boundary of Y in X. A continuum X is said to be *regular* if for each $\varepsilon > 0$, there exists a finite open cover $\mathbf{A}$ of X with mesh$\mathbf{A} < \varepsilon$ such that $\mathrm{Bd}(\mathbf{A})$ is finite. A locally connected continuum is said to be a *dendrite* if it contains no simple closed curve. See [N] for dendrites. It is known that all dendrites and all graphs are regular.

Seidler proved in [S] the following theorem.

Theorem 1.1. *If f is a homeomorphism from a regular continuum X into itself, then the topological entropy $h(f)$ of f is equal to zero.*

This implies that if X is either a dendrite or a graph, then $h(f) = 0$ for all homeomorphism $f : X \to X$.

A continuous map $f : X \to X$ is said to be *monotone* if $f^{-1}(Y)$ is connected for each connected subset Y of $f(X)$. It is well known that f^n is monotone for each positive integer n if f is monotone. It follows from [KS, Theorem D] that if X is either a compact interval or the circle, then $h_\infty(f) = 0$ for all monotone map $f : X \to X$. And Efremova and Markhrova in [EM] considered some class D of dendrites and showed that the topological entropy $h(f)$ of f is equal to zero for all monotone map f from $X \in D$ to itself. It is known from [BC] that $h(f) = h(f|_{\Omega(f)})$ for all continuous map f from a compact space to itself, where $\Omega(f)$ denotes the non-wandering set of f. In fact, it is proved in [EM] that $h(f|_{\Omega(f)}) = 0$ for all monotone map f from $X \in D$ to itself. But by [C2], there exists a continuous map $f : [0,1] \to [0,1]$ such that $h_S(f) > h_S(f|_{\Omega(f)})$ for a suitable sequence S of positive integers. This shows that the topological sequence entropy is different from the topological entropy.

We show the following theorem which is an extension of [KS, Theorem D] and [EM, Theorem B(B4)].

Theorem 1.2. *Let X be either a dendrite or a graph and f a monotone map from X into itself. Then $h_\infty(f) = 0$. In particular, the topological entropy of f is equal to zero.*

2. Definitions.

Definition 2.1. Let Y be a subspace of a metric space X. $\mathrm{Cl}(Y)$ and diam$Y$ denote the closure and the diameter of Y in a space X, respectively.

The cardinality of a set P will be denoted by $\mathrm{Card}(P)$.

Definition 2.2. We say that a cover $\mathbf{B}$ is *finer than* a cover $\mathbf{A}$, and write $\mathbf{B} \geq \mathbf{A}$ if each $B \in \mathbf{B}$ is contained in some $A \in \mathbf{A}$. Clearly, if $\mathbf{B} \geq \mathbf{A}$, then $\bigvee_{i=1}^n f^{-s_i}(\mathbf{B}) \geq \bigvee_{i=1}^n f^{-s_i}(\mathbf{A})$ for any finite sequence $s_1, s_2, \ldots, s_n$ of positive integers, and $N(\mathbf{B}) \geq N(\mathbf{A})$.

Definition 2.3. Let f be a continuous map from a continuum X to itself and n a positive integer. Denote $D_{f,n} = \{x \in X|\mathrm{Card}(f^{-n}(x)) \leq 1\}$ and $D_f = \bigcap_{n=1}^\infty D_{f,n}$. Since $D_{f,n}$ is a G_δ set for each n, we see that D_f is so. A continuum X is said to be *regular for f* if for each $\varepsilon > 0$ there exists a finite open cover $\mathbf{A}$ of X with mesh$\mathbf{A} < \varepsilon$ such that $\mathrm{Bd}(A)$ is finite contained in D_f for each $A \in \mathbf{A}$.

3. Elementary lemmas.

Lemma 3.1. *Let f be a continuous map from a continuum X to itself, $S = \{s_i | i = 1, 2, \dots\}$ an increasing unbounded sequence of positive integers and $\{\mathbf{A}_n\}$ a sequence of finite open covers of X with $\lim_{n\to\infty} \mathrm{mesh}(\mathbf{A}_n) = 0$ and $\mathbf{A}_{n+1} \geq \mathbf{A}_n$ for all n. Then $h_S(f) = \lim_{n\to\infty} h_S(f, \mathbf{A}_n)$.*

Corollary 3.2. Let f be a continuous map from a continuum X to itself, $S = \{s_i | i = 1, 2, \dots\}$ an increasing unbounded sequence of positive integers. If for each $\varepsilon > 0$ there exists a finite open cover of X with $\mathrm{mesh}\mathbf{A} < \varepsilon$ and $h_S(f, \mathbf{A}) = 0$, then $h_S(f) = 0$.

Lemma 3.3. *Let f be a continuous map from a continuum X to itself, n a positive integer, $\mathbf{A}$ an open cover of X with $\mathrm{Card}(\mathbf{A}) \geq 2$, $s_1, s_2, \dots, s_{n-1}$ a sequence of positive integers and $\mathbf{B}$ a subcover of $\bigvee_{i=1}^{n-1} f^{-s_i}(\mathbf{A})$. Then $\mathrm{Bd}(\mathbf{B}) \subset \mathrm{Bd}(\bigcup_{i=1}^{n-1} f^{-s_i}(\mathbf{A}))$.*

Lemma 3.4. *Let f be a continuous map from a continuum X to itself and $\mathbf{A}$ a finite open cover of X with $\mathrm{Card}(\mathbf{A}) \geq 2$ such that $\mathrm{Bd}(\mathbf{A})$ is finite contained in D_f. And let $L_{\mathbf{A}} = \sum_{A\in\mathbf{A}} \mathrm{Card}(\mathrm{Bd}(\mathrm{A}))$. Then $N(\bigvee_{i=1}^{n-1} f^{-s_i}(\mathbf{A})) \leq nL_{\mathbf{A}}$ for any sequence $s_1, s_2, \dots, s_{n-1}$ of positive integers.*

Lemma 3.5. *Let f be a monotone map from a regular continuum X to itself. Then D_f is a G_δ dense set in X.*

4. A proof of Theorem 1.2.

Theorem 4.1. *Let f be a monotone map from X into itself. If X is regular for f, then $h_\infty(f) = 0$. In particularly, $h(f) = 0$.*

Corollary 4.2. Let f be a continuous map from a regular continuum X to itself. If f is embedding, then $h_\infty(f) = 0$.

Proof of Theorem 1.2. Let X be a dendrite or a graph and f a monotone map from X to itself. Lemma 3.5 implies D_f is a G_δ dense set. Let $\varepsilon > 0$. Since X is a dendrite or a graph, there exists a finite set $F \subset D_f$ such that $\mathrm{diam}B < 1/2\varepsilon$ for all $B \in \mathbf{B}$, where $\mathbf{B}$ is the set of all components of $X \setminus F$. Set $A_x = \bigcup\{\{x\} \cup B | B \in \mathbf{B}$ and $x \in \mathrm{Cl}(B)\}$ for each $x \in F$ and $\mathbf{A} = \{A_x | x \in F\}$. We see that $\mathbf{A}$ is a finite open cover of X with $\mathrm{mesh}\mathbf{A} < \varepsilon$ such that $\mathrm{Bd}(\mathbf{A}) \subset F \subset D_f$. This shows that X is regular for f. It follows from Theorem 4.1 that $h_\infty(f) = 0$. □

From the proof of Theorem 1.2, we have the following corollary.

Corollary 4.3. Let f be a continuous map from a regular continuum X to itself. If for each $\varepsilon > 0$ there exists a finite subset F of D_f such that $\mathrm{diam}B < \varepsilon$ for all component B of $X \setminus F$, then $h_\infty(f) = 0$.

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Institute of Mathematics, University of Tsukuba, Ibraki 305-8571 Japan
E-mail address: naochin@math.tsukuba.ac.jp

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HEREDITARILY SEPARABLE GROUP TOPOLOGIES ON ABELIAN GROUPS

DIKRAN DIKRANJAN AND DMITRI SHAKHMATOV

Dedicated to the memory of Zoltán T. Balogh

All group topologies in this paper are considered to be Hausdorff (and thus Tychonoff). Recall that a topological space X is:

Lindelöf if every open cover of X has a countable subcover,

(countably) compact if every (countable) open cover of X has a finite subcover,

pseudocompact if every real-valued continuous function defined on X is bounded, and

separable if X has a countable dense subset.

It is well-known that compact $\rightarrow$ countably compact $\rightarrow$ pseudocompact, and "pseudocompact + Lindelöf" $\leftrightarrow$ compact.

Recall that a topological group G is *precompact*, or *totally bounded*, if G is (topologically and algebraically isomorphic to) a subgroup of some compact group. Pseudocompact groups are precompact [7], so we have a somewhat longer chain

$$\text{compact} \rightarrow \text{countably compact} \rightarrow \text{pseudocompact} \rightarrow \text{precompact}$$

of compactness-like conditions for topological groups.

A space X is called *hereditarily separable* if every subspace of X is separable (in the subspace topology), and X is said to be *hereditarily Lindelöf* if every subspace of X is Lindelöf (in the subspace topology). An S*-space* is a hereditarily separable regular space that is not Lindelöf.

1. Motivation

Our results originate in three diverse areas of mathematics.

The first source of inspiration comes from the celebrated theory of S-spaces in set-theoretic topology, and especially, a famous 1975 example of Fedorčuk of a hereditarily separable compact space of size $2^{\mathfrak{c}}$. In our paper we completely characterize Abelian groups that admit a group topology making them into an S-space, and we produce the "best possible analogues" of the Fedorčuk space in the

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category of topological groups. As it turns out, a vast majority of Abelian groups admit group topologies with properties similar to that of the Fedorčuk example.

The second origin lies in topological algebra, where we were motivated by the problem of which Abelian groups admit a countably compact group topology. We completely describe, albeit consistently, the algebraic structure of Abelian groups of size at most $2^{\mathfrak{c}}$ that admit a countably compact group topology.

Our third motivation comes from the theory of cardinal invariants in general topology. We resolve completely a 1980 problem of van Douwen about the cofinality of $|G|$ for a countably compact group G in the case of Abelian groups.

We will now address all three sources of our motivation in detail.

1.1. *S*-groups à la Fedorčuk. Recall that $|Y| \leq \mathfrak{c}$ for a hereditarily Lindelöf Hausdorff space Y [1], and $|X| \leq 2^{\mathfrak{c}}$ for a separable Hausdorff space X [32]. It is natural to ask whether the last inequality can be strengthened to $|X| \leq \mathfrak{c}$ for a *hereditarily* separable regular space X. If there are no S-spaces, then every hereditarily separable regular space X is hereditarily Lindelöf, and therefore $|X| \leq \mathfrak{c}$ by the result cited above. Todorčević has proved the consistency with ZFC that S-spaces do *not* exist ([40], see also [41]). Therefore, in Todorčević's model of ZFC, hereditarily separable regular spaces have size at most $\mathfrak{c}$. A first consistent example of a hereditarily separable Tychonoff space of size $2^{\mathfrak{c}}$ has been found by Hajnal and Juhász [20]. Two years later Fedorčuk [17] produced the strongest known example up to date using his celebrated inverse spectra with fully closed maps (see also [18]):

Example 1.1. The existence of the following "Fedorčuk space" X is consistent with ZFC plus CH:
(i) $|X| = 2^{\mathfrak{c}}$,
(ii) X is hereditarily separable,
(iii) X is compact, and
(iv) if F is an infinite closed subset of X, then $|F| = |X|$; in particular, X does not contain non-trivial convergent sequences.

The main goal of this paper is to address the question of the existence of "Fedorčuk space" in the context of topological groups. That is, given a group G, we wonder if it is possible to find a hereditarily separable Hausdorff group topology on G having properties that "Fedorčuk space" has. Since we want to get a hereditarily separable topology on G, we have to restrict ourselves to groups G of size at most $2^{\mathfrak{c}}$. One naturally expects that the presence of algebra may produce additional restrictions on how good a Fedorčuk type group can be. And this is indeed the case.

First of all, one is forced to relax somewhat the compactness condition from item (iii) of Example 1.1 because of two fundamental facts about compact groups:

Fact 1.2. *(i) Infinite compact groups contain non-trivial convergent sequences.*
(ii) Compact hereditarily separable groups are metrizable.

Both facts are folklore and follow from the following result of Hagler, Gerlits and Efimov: An infinite compact group G contains a copy of the Cantor cube $\{0,1\}^{w(G)}$, where $w(G)$ is the weight of G. An elementary proof of this theorem, together with some historical discussion, can be found in [35].

Recall that a space X is *initially* ω_1*-compact* if every open cover of size $\leq \omega_1$ has a finite subcover. Item (i) of Fact 1.2 is no longer valid, at least consistently, if one replaces “compact” by “initially ω_1-compact” in it: It is consistent with ZFC that there exists an initially ω_1-compact Hausdorff group topology without non-trivial convergent sequences on the free Abelian group of size $\mathfrak{c}$. This result is announced, with a hint at a proof, in [42].

However, item (ii) of Fact 1.2 remains valid if one replaces “compact” by “initially ω_1-compact” in it, see [2]. This means that countable compactness appears to be the *strongest* compactness type property among weakenings of classical compactness for which one may hope to obtain hereditarily separable group topologies, and indeed, consistent examples of hereditarily separable countably compact groups (without non-trivial convergent sequences) are known in the literature [21, 38, 28]. This perfectly justifies countable compactness as our strongest compactness condition of choice when working with hereditarily separable groups.

Second, we will have to restrict ourselves to Abelian groups because in the non-commutative case there are groups (of small size) that do not admit *any* countably compact or separable group topology, as follows from our next result:

Proposition 1.3. *Let X be a set and $S(X)$ the symmetric group of X.*[1] *Then:*
(i) $S(X)$ does not admit a separable group topology unless X is countable,
(ii) $S(X)$ admits no countably compact group topology when X is infinite, and
(iii) $S(X)$ does not admit a Lindelöf group topology unless X is countable.

Proof. We equip $S(X)$ with the topology of pointwise convergence on X, i.e. the topology $\mathcal{T}_p$ generated by the family $\{U(f,F) : f \in S(X), F \in [X]^{<\omega}\}$ as a base, where $U(f,F) = \{g \in S(X) : g(x) = f(x) \text{ for all } x \in F\}$. It is easy to see that $\mathcal{T}_p$ is a group topology.

Assume that X is an infinite set. For a fixed $x \in X$, the stabilizer $S_x = \{\sigma \in S(X) : \sigma(x) = x\} = U(id_X, \{x\})$ of x is a $\mathcal{T}_p$-open subgroup of $S(X)$ of index $|X|$, and hence it produces an open cover of $S(X)$ by pairwise disjoint sets (obtained by taking appropriate unions of cosets of S_x) without a subcover of size (strictly) less than $|X|$. It follows that the space $(S(X), \mathcal{T}_p)$ is not countably compact, and also is neither separable nor Lindelöf when $|X| > \omega$.

It is known that $\mathcal{T}_p$ is a minimal element in the lattice of all (Hausdorff) group topologies on $S(X)$, i.e. $\mathcal{T}_p \subseteq \mathcal{T}$ for every (Hausdorff) group topology $\mathcal{T}$ on $S(X)$ [19]. This easily yields the conclusion of all three items of our proposition. □

It follows from the above proposition that, for an uncountable set X, the symmetric group $S(X)$ admits neither a separable, nor a countably compact, nor a Lindelöf group topology.[2] Furthermore, free groups never admit countably compact group topologies ([10, Theorem 4.7]; see also [12, Corollary 5.14]).

Third, algebraic restrictions prevent us from getting the full strength of item (iv), as our next example demonstrates:

[1] That is, $S(X)$ is a set of bijections of X onto itself with the composition of maps as multiplication.

[2] In particular, no group $S(X)$ admits a Hausdorff *group* topology that makes it into an S-space. This should be compared with substantial difficulties one has to overcome to produce a model of ZFC in which there are no *S-spaces*. Furthermore, no group $S(X)$ admits a Hausdorff *group* topology that makes it into an L-space (i.e., a hereditarily Lindelöf but not (hereditarily) separable space). This should be compared with the fact that the consistency of the non-existence of *L-spaces* is a well-known problem of set-theoretic topology that remains unresolved.

Example 1.4. Let $G = \mathbb{Z}(2)^{(\mathfrak{c})} \oplus \mathbb{Z}^{(2^{\mathfrak{c}})}$ be the direct sum of the Boolean group $\mathbb{Z}(2)^{(\mathfrak{c})}$ of size $\mathfrak{c}$ and the free Abelian group $\mathbb{Z}^{(2^{\mathfrak{c}})}$ of size $2^{\mathfrak{c}}$. We claim that, *for any Hausdorff group topology on G, there exists a closed (in this topology) infinite set F such that $|F| < |G|$.* In fact, $F = \mathbb{Z}(2)^{(\mathfrak{c})} \subseteq G$ is such a set. Indeed, $|F| = \mathfrak{c} < 2^{\mathfrak{c}} = |G|$, so it remains only to note that F is an *unconditionally closed* subset of G in Markov's sense [31]; that is, F is closed in every Hausdorff group topology on G. The latter follows from the fact that $F = \{x \in G : 2x = 0\}$ is the preimage of the (closed!) set $\{0\}$ under the continuous map that sends x to $2x$.

We note that our Theorem 2.7 implies that, in an appropriate model of ZFC, the group G from the example above does admit a hereditarily separable countably compact group topology without non-trivial convergent sequences. So the best we can hope for in our quest for Fedorčuk type group G is to require that G satisfies the second, weaker, condition from item (iv) of Example 1.1, i.e. that G *does not have any non-trivial convergent sequences*. In fact, we will manage to get a stronger condition: G *does not have infinite compact subsets.*

1.2. Algebraic structure of countably compact Abelian groups.

Halmos [22] showed that the additive group of real numbers can be equipped with a compact group topology and asked which Abelian groups admit compact group topologies. Halmos' problem seeking a complete description of the algebraic structure of compact Abelian groups contributed substantially to the development of the Abelian group theory, particularly through the introduction of the algebraically compact groups by Kaplansky [26]. This problem has been completely solved in [23, 24].

The counterpart of Halmos' problem for pseudocompact groups asking which Abelian groups can be equipped with a pseudocompact group topology was attacked in [3, 10, 11, 4, 5, 12] and the significant progress has been summarized in the monograph [12]. Recall also that every Abelian group admits a precompact group topology [6].

The question of which Abelian groups admit a countably compact group topology appears to be much more complicated. After a series of scattered results [21, 15, 38, 28, 13, 43] a complete description of the algebraic structure of countably compact Abelian groups of size at most $\mathfrak{c}$ under Martin's Axiom MA has been recently obtained in [14]: MA implies that an Abelian group G of size at most $\mathfrak{c}$ admits a countably compact group topology if and only if it satisfies both **PS** and **CC**, two conditions introduced in Definition 2.3 below. (In particular, every torsion-free Abelian group of size $\mathfrak{c}$ admits a countably compact group topology under MA [39].) In our Theorem 2.7 and Corollary 2.17(ii) we substantially extend this result by proving that, at least consistently, the conjunction of **PS** and **CC** is both a necessary and a sufficient condition for the existence of a countably compact group topology on an Abelian group G of size at most $2^{\mathfrak{c}}$. Moreover, we get both hereditary separability and absence of infinite compact subsets for our group topology as a bonus.

This "jump" from $\mathfrak{c}$ to $2^{\mathfrak{c}}$ is an essential step forward. Indeed, amazingly little is presently known about the existence of countably compact group topologies on groups of cardinality greater than $\mathfrak{c}$. Using a standard closing-off argument van Douwen [16] showed that every infinite Boolean group of size $\kappa = \kappa^{\omega}$ admits a countably compact group topology and his argument can easily be extended to

Abelian groups of prime exponent. It is consistent with ZFC that the Boolean group of size κ has a countably compact group topology provided that $\mathfrak{c} \leq \kappa \leq 2^{\mathfrak{c}}$ [44]. (Here $2^{\mathfrak{c}}$ can be made "arbitrary large".) It is also consistent with ZFC that the free Abelian group of size κ has a countably compact group topology provided that $\mathfrak{c} \leq \kappa = \kappa^{\omega} \leq 2^{\mathfrak{c}}$ [27]. Finally, it is well-understood which Abelian groups admit compact group topologies. Essentially these are the *only* known results in the literature about the existence of countably compact group topologies on groups of cardinality greater than $\mathfrak{c}$ (even without the additional requirement of hereditary separability).

While the algebraic description of Abelian groups admitting either a compact or a pseudocompact group topology can be carried out without any additional set-theoretic assumptions beyond ZFC, all known results about countably compact topologizations described above have either been obtained by means of some additional set-theoretic axioms (usually Continuum Hypothesis CH or versions of Martin's Axiom MA) or their consistency has been proved by forcing. Even the fundamental question (raised in [38]) as to whether the free Abelian group of size $\mathfrak{c}$ admits a countably compact group topology is still open in ZFC. (Recall that no free Abelian group admits a compact group topology.)

It seems worth noting a peculiar difference between compact and countably compact topologizations of Abelian groups. In the compact case the sufficiency of the algebraic conditions is relatively easy to prove, whereas their necessity is much harder to establish. In the countably compact case the necessity of **PS** and **CC** is immediate (see Lemma 2.5), while the sufficiency is rather complicated and at the present stage requires additional set-theoretic assumptions.

1.3. van Douwen's problem: Is $|G| = |G|^{\omega}$ for a countably compact group G? It is well-known that $|G| = 2^{w(G)}$ for an infinite compact group G, where $w(G)$ is the weight of G [25]. In particular, the cardinality $|G|$ of an infinite compact group G satisfies the equation $|G| = |G|^{\omega}$. This motivated van Douwen to ask in [16] the following natural question: Does $|G| = |G|^{\omega}$, or at least $\mathrm{cf}(|G|) > \omega$, hold for every infinite topological group (or homogeneous space) G which is *countably* compact?

In the same paper [16] van Douwen proved that, under the Generalized Continuum Hypothesis GCH, every infinite pseudocompact homogeneous space G satisfies $|G| = |G|^{\omega}$. In particular, a strong positive answer (with countable compactness weakened to pseudocompactness, and "topological group" weakened to "homogeneous space") to van Douwen's problem is consistent with ZFC. A first consistent counter-example to van Douwen's question was recently announced by Tomita [44] who used forcing to construct a model of ZFC in which every Boolean group of size κ has a countably compact group topology provided that $\mathfrak{c} \leq \kappa \leq 2^{\mathfrak{c}}$ [44, Theorem 2.2]. Here $2^{\mathfrak{c}}$ can be made "arbitrary large" so that, for any given ordinal $\sigma \geq 1$ chosen in advance, one can arrange that $\mathfrak{c} \leq \aleph_{\sigma} \leq 2^{\mathfrak{c}}$ (in particular, $\aleph_{\omega}$ can be included in the interval between $\mathfrak{c}$ and $2^{\mathfrak{c}}$).

In our Corollary 2.23 we push Tomita's negative solution to van Douwen's question to the extreme limit by demonstrating that, in a sense, the cofinality of $|G|$ for a countably compact Abelian group G is completely irrelevant: For every ordinal $\sigma \geq 1$ it is consistent with ZFC that *every* Abelian group G of size $\aleph_{\sigma}$ admits a countably compact group topology provided that G satisfies **PS** and **CC**, two necessary conditions for the existence of such a topology on G (see Definition 2.3 and Lemma 2.5(ii)).

2. Main results

The major achievement of this paper is a forcing construction of a (class of) special model(s) of ZFC in which Abelian groups of size at most $2^{\mathfrak{c}}$ admit hereditarily separable group topologies with various compactness-like properties and without infinite compact subsets. Let us produce an outline of our construction.

We define, for every cardinal $\kappa \geq \omega_2$, a forcing notion $(\mathbb{P}_\kappa, \leq)$ that depends only on this cardinal κ (thereby justifying the notation $\mathbb{P}_\kappa$). Let M_κ be an arbitrary countable transitive model of ZFC satisfying $\kappa \in M_\kappa$, $(\mathbb{P}_\kappa, \leq) \in M_\kappa$, $\mathfrak{c} = \omega_1$ and $2^{\omega_1} = \kappa$. If $\mathbb{G} \subseteq \mathbb{P}_\kappa$ is a $\mathbb{P}_\kappa$-generic set over M_κ, then the generic extension $M_\kappa[\mathbb{G}]$ has the same cardinals as M_κ and the equalities $\omega_1 = \mathfrak{c}$ and $2^{\omega_1} = 2^{\mathfrak{c}} = \kappa$ hold in $M_\kappa[\mathbb{G}]$. Since the original cardinal κ can be taken to be "arbitrarily large", the power $2^{\mathfrak{c}}$ of the continuum $\mathfrak{c}$ in the generic extension $M_\kappa[\mathbb{G}]$ can also be made "arbitrarily large".

In all results below, $M_\kappa[\mathbb{G}]$ will always denote the generic extension described above.

Our first main result shows that, at least consistently, the inequality $|G| \leq 2^{\mathfrak{c}}$ is the only necessary condition for the existence of a hereditarily separable group topology on an Abelian group:

Theorem 2.1. *In $M_\kappa[\mathbb{G}]$, the following conditions are equivalent for any Abelian group G:*

(i) G admits a separable group topology,

(ii) G admits a hereditarily separable group topology,

(iii) G admits a hereditarily separable precompact group topology without infinite compact subsets, and

(iv) $|G| \leq 2^{\mathfrak{c}}$.

Recall that Todorčević constructed a model of ZFC in which S-spaces do not exist ([40], see also [41]). Things change dramatically in this model:

Theorem 2.2. *In any model of ZFC in which there are no S-spaces the following conditions are equivalent for any Abelian group G:*

(i) G admits a hereditarily separable group topology,

(ii) G admits a separable metric precompact group topology, and

(iii) $|G| \leq \mathfrak{c}$.

We would like to emphasize that there are absolutely no algebraic restrictions (except natural restriction of commutativity) on the group G in the above two theorems. Algebraic constraints become more prominent when one adds some compactness condition to the mix.

Let G be an Abelian group. As usual $r(G)$ denotes the *free rank* of G. For every natural number $n \geq 1$ define $G[n] = \{g \in G : ng = 0\}$ and $nG = \{ng : g \in G\}$. Recall that G is:

torsion provided that $G = \bigcup\{G[n] : n \in \omega \setminus \{0\}\}$,
bounded torsion if $G = G[n]$ for some $n \in \omega \setminus \{0\}$,
torsion-free if $G[n] = \{0\}$ for every $n \in \omega \setminus \{0\}$, and
divisible if $mG = G$ for each $m \in \omega \setminus \{0\}$.

We will now introduce three algebraic conditions that will play a prominent role throughout this paper.

Definition 2.3. For an Abelian group G, define the following three conditions:

PS: Either $r(G) \geq \mathfrak{c}$ or G is a bounded torsion group.

CC: For every pair of integers $n \geq 1$ and $m \geq 1$ the group $mG[n]$ is either finite or has size at least $\mathfrak{c}$.

tCC: If G is torsion, then **CC** holds.

Our next lemma, despite its simplicity, is quite helpful for better understanding of these conditions:

Lemma 2.4. *Let G be an Abelian group.*

(i) If G is torsion, then G satisfies **PS** *if and only if G is a bounded torsion group.*

(ii) If G is a torsion-free group, then G satisfies **PS** *if and only if $|G| \geq \mathfrak{c}$.*

(iii) If G is a torsion-free group, then G satisfies **CC**.

(iv) **CC** *for G implies* **tCC**.

(v) If G is not torsion, then G satisfies **tCC**.

(vi) If G is torsion and satisfies **tCC**, *then G satisfies* **CC** *as well.*

Proof. To prove (i) note that $r(G) = 0 < \mathfrak{c}$ if G is torsion.

(ii) If G is a torsion-free group, then condition **PS** for G becomes equivalent to $r(G) \geq \mathfrak{c}$, and the latter condition is known to be equivalent to $|G| \geq \mathfrak{c}$.

(iii) Assume that G is torsion-free. Let $n \geq 1$ and $m \geq 1$ be natural numbers. Then $G[n] = \{0\}$ and hence $mG[n] = \{0\}$ is finite. Therefore **CC** holds.

Items (iv), (v) and (vi) are trivial. □

Condition **PS** is known to be necessary for the existence of a pseudocompact group topology on an Abelian group G, thereby justifying its name (**PS** stands for "pseudocompact"). To the best of the author's knowledge, this fact has been announced without proof in [3, Remark 2.17] and [10, Proposition 3.3], and has appeared in print with full proof in [12, Theorem 3.8].

It can be easily seen that condition **CC** is necessary for the existence of a countably compact group topology on an Abelian group G, thereby justifying its name (**CC** stands for "countably compact"). Indeed, if G is a countably compact group, then the set $G[n] = \{g \in G : ng = 0\}$ must be closed in G, and thus $G[n]$ is countably compact in the subspace topology induced on $G[n]$ from G. Furthermore, the map which sends $g \in G[n]$ to $mg \in mG[n]$ is continuous, and so $mG[n]$ must be countably compact (in the subspace topology). It remains only to note that an infinite countably compact group has size at least $\mathfrak{c}$ [16, Proposition 1.3 (a)]. In the particular case when an Abelian group G has size $\mathfrak{c}$, the fact that **CC** is a necessary condition for the existence of a countably compact group topology on G has been proved in [14].

Condition **CC** has essentially appeared for the first time in [10] where it was proved that **CC** is necessary for the existence of a pseudocompact group topology on a *torsion* Abelian group.[3] Since **CC** and **tCC** are equivalent for torsion groups by items (v) and (vii) of Lemma 2.4, it follows that **tCC** is a necessary condition

[3]Furthermore, it is proved in [10] that **CC** is also a sufficient condition for the existence of a pseudocompact group topology on a *bounded* torsion Abelian group of size at most $2^{\mathfrak{c}}$. See also the proof of Theorem 2.22.

for the existence of a pseudocompact group topology on a torsion group, thereby justifying our choice of terminology (**tCC** stands for "torsion **CC**"). Since **tCC** trivially holds for non-torsion groups (see item (vi) of Lemma 2.4), we conclude that **tCC** is a necessary condition for the existence of a pseudocompact group topology on an Abelian group G.

We can now summarize the discussion above in a convenient lemma:

Lemma 2.5. *(i) A pseudocompact Abelian group G satisfies* **PS** *and* **tCC***.*
(ii) A countably compact Abelian group G satisfies **PS** *and* **CC***.*

In the "opposite direction", it is known that the combination of **PS** and **tCC** is sufficient for the existence of a pseudocompact group topology on an Abelian group G of size at most $2^{\mathfrak{c}}$ ([10]; see also [12]) and, under Martin's Axiom MA, the combination of **PS** and **CC** is sufficient for the existence of a countably compact group topology on an Abelian group G of size at most $\mathfrak{c}$ [14].

In our next "twin" theorems we establish that these pairs of conditions are, consistently, also sufficient for the existence of a *hereditarily separable* pseudocompact and countably compact group topology on a group G of size at most $2^{\mathfrak{c}}$.

Theorem 2.6. *In $M_\kappa[\mathbb{G}]$, the following conditions are equivalent for any Abelian group G:*
(i) G admits a separable pseudocompact group topology,
(ii) G admits a hereditarily separable pseudocompact group topology,
(iii) G admits a hereditarily separable pseudocompact group topology without infinite compact subsets, and
(iv) $|G| \le 2^{\mathfrak{c}}$ and G satisfies both **PS** *and* **tCC***.*

We can also prove that the equivalence of items (i) and (iv) in the above theorem holds in ZFC.

Theorem 2.7. *In $M_\kappa[\mathbb{G}]$, the following conditions are equivalent for any Abelian group G:*
(i) G admits a separable countably compact group topology,
(ii) G admits a hereditarily separable countably compact group topology,
(iii) G admits a hereditarily separable countably compact group topology without infinite compact subsets, and
(iv) $|G| \le 2^{\mathfrak{c}}$ and G satisfies both **PS** *and* **CC***.*

Theorem 2.7 recovers (and greatly extends) the main result of Dikranjan and Tkachenko [14]: It is consistent with ZFC that an Abelian group G of size at most $\mathfrak{c}$ has a countably compact group topology if and only if G satisfies both **PS** and **CC**.

Things become "essentially trivial" in Todorčević's model of ZFC without S-spaces:

Theorem 2.8. *In any model of ZFC in which there are no S-spaces the following conditions are equivalent for any Abelian group G:*
(i) G admits a hereditarily separable pseudocompact group topology,
(ii) G admits a hereditarily separable countably compact group topology, and
(iii) G admits a compact metric group topology.

Let G be any Abelian group such that $\mathfrak{c} < |G| \le 2^{\mathfrak{c}}$. Since compact metric spaces have size at most $\mathfrak{c}$, our previous theorem implies that, consistently, G does not admit a hereditarily separable pseudocompact group topology. On the other hand, if one additionally assumes that G satisfies both **PS** and **CC**, then G admits a hereditarily separable countably compact group topology in the model $M_\kappa[\mathbb{G}]$ (Theorem 2.7). In particular, we conclude that *the existence of a hereditarily separable pseudocompact (or countably compact) group topology on the free Abelian group of size* $2^{\mathfrak{c}}$ *is both consistent with and independent of ZFC*. (An example of an Abelian group of size $\mathfrak{c}$ with similar properties is much harder to obtain.)

We will now look at what our Theorems 2.6 and 2.7 say for four particular important subclasses of Abelian groups: torsion groups, non-torsion groups, torsion-free groups, and divisible groups.

Corollary 2.9. *In* $M_\kappa[\mathbb{G}]$*, the following conditions are equivalent for any torsion Abelian group* G*:*

(i) G *admits a separable pseudocompact group topology,*

(ii) G *admits a hereditarily separable countably compact group topology without infinite compact subsets, and*

(iii) $|G| \leq 2^{\mathfrak{c}}$ *and* G *is a bounded torsion group satisfying* **CC**.

Proof. Let G be a torsion Abelian group. According to Lemma 2.4(i), a bounded torsion group satisfies **PS**, so (iii) implies (ii) by Theorem 2.7. The implication (ii) $\to$ (i) is trivial. To see that (i) $\to$ (iii), note that $|G| \leq 2^{\mathfrak{c}}$ and G satisfies both **PS** and **tCC** by Lemma 2.5(i). Since G is torsion, Lemma 2.4(i) yields that G is a bounded torsion group, while Lemma 2.4(vi) implies that G satisfies **CC**. □

The following particular case of the above corollary seems to be worth mentioning:

Corollary 2.10. *In* $M_\kappa[\mathbb{G}]$*, for every prime number* p*, each natural number* $n \geq 1$ *and every infinite cardinal* τ*, the following conditions are equivalent:*

(i) $\mathbb{Z}(p^n)^{(\tau)}$ *admits a separable pseudocompact group topology,*

(ii) $\mathbb{Z}(p^n)^{(\tau)}$ *admits a hereditarily separable countably compact group topology without infinite compact subsets, and*

(iii) $\mathfrak{c} \leq \tau \leq 2^{\mathfrak{c}}$.

Proof. For the group $\mathbb{Z}(p^n)^{(\tau)}$, condition **CC** is equivalent to "τ is either finite or $\tau \geq \mathfrak{c}$", and the result follows from Corollary 2.9. □

Since torsion pseudocompact groups are always zero-dimensional [9], the assumption that G is non-torsion is necessary in the next two theorems.

Theorem 2.11. *In* $M_\kappa[\mathbb{G}]$*, the following conditions are equivalent for any non-torsion Abelian group* G*:*

(i) G *admits a separable pseudocompact group topology,*

(ii) G *admits a hereditarily separable connected and locally connected pseudocompact group topology without infinite compact subsets, and*

(iii) $|G| \leq 2^{\mathfrak{c}}$ *and* G *satisfies* **PS**.

Theorem 2.12. *In* $M_\kappa[\mathbb{G}]$*, the following conditions are equivalent for any non-torsion Abelian group* G*:*

(i) G *admits a separable countably compact group topology,*

(ii) G *admits a hereditarily separable connected and locally connected countably compact group topology without infinite compact subsets, and*

(iii) $|G| \leq 2^{\mathfrak{c}}$ *and* G *satisfies* **PS** *and* **CC**.

In the case of torsion-free groups things become very transparent, as algebraic restraints disappear again:

Corollary 2.13. *In $M_\kappa[\mathbb{G}]$, the following conditions are equivalent for any torsion-free Abelian group G:*

(i) G admits a separable pseudocompact group topology,

(ii) G admits a hereditarily separable countably compact connected and locally connected group topology without infinite compact subsets, and

(iii) $\mathfrak{c} \leq |G| \leq 2^{\mathfrak{c}}$.

Proof. Let G be a torsion-free Abelian group. According to item (iii) of Lemma 2.4, condition **PS** for G is equivalent to $|G| \geq \mathfrak{c}$, while items (iv) and (v) of the same lemma imply that both conditions **CC** and **tCC** hold for G. It remains only to plug these facts into Theorems 2.11 and 2.12. □

We note that even a very particular case of our Corollary 2.13 constitutes the main result of Koszmider, Tomita and Watson [27]: It is consistent with ZFC that for every cardinal τ such that $\mathfrak{c} \leq \tau = \tau^\omega \leq 2^{\mathfrak{c}}$ the free Abelian group of size τ admits a countably compact group topology without non-trivial convergent sequences. The topology constructed in [27] is not hereditarily separable, while our topology is. Furthermore, while our topology does not have infinite compact subsets, it is not at all clear if the topology from [27] has infinite compact subsets or not.

As usual, for a prime number p and an Abelian group G, $r_p(G)$ denotes the *p-rank* of G. Our next theorem reduces the problem of the existence of a (hereditarily) separable countably compact group topology on a divisible Abelian group G to a simple checking of transparent conditions involving the cardinality, free rank and p-ranks of G.

Theorem 2.14. *In $M_\kappa[\mathbb{G}]$, the following conditions are equivalent for any non-trivial divisible Abelian group G:*

(i) G admits a separable countably compact group topology,

(ii) G admits a hereditarily separable connected and locally connected countably compact group topology without infinite compact subsets,

(iii) $\mathfrak{c} \leq r(G) \leq |G| \leq 2^{\mathfrak{c}}$ and, for every prime number p, either the p-rank $r_p(G)$ of G is finite or the inequality $r_p(G) \geq \mathfrak{c}$ holds.

Corollary 2.15. *In $M_\kappa[\mathbb{G}]$, the following conditions are equivalent for any Abelian group G:*

(i) G admits a separable connected precompact group topology,

(ii) G admits a hereditarily separable connected and locally connected pseudocompact group topology without infinite compact subsets.

Proof. (i) →(ii). Since G is precompact, there exists a non-trivial continuous character $\chi : G \to \mathbb{T}$. Then $\chi(G)$ is a non-trivial connected subgroup of $\mathbb{T}$, which yields $\chi(G) = \mathbb{T}$. Therefore $r(G) \geq r(\mathbb{T}) = \mathfrak{c}$. In particular, G is non-torsion and satisfies **PS**. The separability of G yields $|G| \leq 2^{\mathfrak{c}}$. Now implication (iii)→(ii) of Theorem 2.11 guarantees that G admits a hereditarily separable connected and locally connected pseudocompact group topology without infinite compact subsets.

(i) →(ii) is trivial. □

Fact 1.2(i) inspired a quest for constructing compact-like group topologies without non-trivial convergent sequences, see, for example, [36, 21, 15, 28, 30, 38, 8, 44].

Our next corollary shows that, in a certain sense, one does not need to work that hard in order to get these topologies: Indeed, at least on Abelian groups of size at most $2^{\mathfrak{c}}$, there are "plenty" of them around in the model $M_\kappa[\mathbb{G}]$:

Corollary 2.16. *In $M_\kappa[\mathbb{G}]$, let G be an Abelian group of size at most $2^{\mathfrak{c}}$. Then:*

(i) G admits a hereditarily separable precompact group topology without infinite compact subsets,

(ii) if G admits a pseudocompact group topology, then G also has a hereditarily separable pseudocompact group topology without infinite compact subsets,

(iii) if G admits a countably compact group topology, then G also has a hereditarily separable countably compact group topology without infinite compact subsets.

Proof. Item (i) follows from the implication (iv) → (iii) of Theorem 2.1. Item (ii) follows from Lemma 2.5(i) and the implication (iv) → (iii) of Theorem 2.6. Item (iii) follows from Lemma 2.5(ii) and the implication (iv) → (iii) of Theorem 2.7. □

As a by-product of our results, we can completely describe the algebraic structure of the Abelian groups of size at most $2^{\mathfrak{c}}$ which admit, at least consistently, a countably compact group topology.

Corollary 2.17. *In $M_\kappa[\mathbb{G}]$, let G be an Abelian group of size at most $2^{\mathfrak{c}}$. Then G admits a countably compact group topology if and only if G satisfies both* **PS** *and* **CC**.

Proof. The "only if" part follows from Lemma 2.5(ii), and the "if" part follows from the implication (iv) → (iii) of Theorem 2.7. □

Corollary 2.18. *In $M_\kappa[\mathbb{G}]$, a torsion Abelian group G of size at most $2^{\mathfrak{c}}$ admits a countably compact group topology if and only if G is bounded and satisfies* **CC**.

Proof. The "only if" part follows from Lemma 2.5(ii), and the "if" part follows from the implication (iii) → (ii) of Corollary 2.9. □

Corollary 2.19. *In $M_\kappa[\mathbb{G}]$, a torsion-free Abelian group G of size at most $2^{\mathfrak{c}}$ admits a countably compact group topology if and only if $|G| \geq \mathfrak{c}$.*

Proof. Corollary 2.13 applies. □

Corollary 2.20. *In $M_\kappa[\mathbb{G}]$, the following two conditions are equivalent for every Abelian group G of size at most $2^{\mathfrak{c}}$ that is either torsion or torsion-free:*

(i) G admits a pseudocompact group topology, and

(ii) G admits a countably compact group topology.

Proof. Clearly (ii) implies (i). To prove the converse, assume (i). Then G satisfies **PS** and **tCC** by Lemma 2.5(i). If G is torsion, G satisfies **CC** by item (vii) of Lemma 2.4. If G is torsion-free, then G satisfies **CC** by item (iv) of Lemma 2.5. Since $|G| \leq 2^{\mathfrak{c}}$ and G satisfies both **PS** and **CC**, Theorem 2.7 now yields that G has a countably compact group topology. □

Corollary 2.21. *In $M_\kappa[\mathbb{G}]$, a divisible Abelian group G of size at most $2^{\mathfrak{c}}$ admits a countably compact group topology if and only if $r(G) \geq \mathfrak{c}$ and, for every prime number p, either the p-rank $r_p(G)$ of G is finite or the inequality $r_p(G) \geq \mathfrak{c}$ holds.*

Proof. This immediately follows from Theorem 2.14. □

The counterpart of Corollary 2.17 for pseudocompact group topologies can be proved in ZFC.

Theorem 2.22. *Let G be an Abelian group of size at most $2^{\mathfrak{c}}$. Then G admits a pseudocompact group topology if and only if G satisfies both* **PS** *and* **tCC**.

We will now exhibit an application of Theorem 2.7 to van Douwen's problem, see Subsection 1.3. Our next corollary demonstrates that, contrary to van Douwen's belief, it is consistent with ZFC that there is nothing exceptional about Abelian groups whose size has countable cofinality, such as $\aleph_\omega$, $\aleph_{\omega+\omega}$, $\aleph_{\omega+\omega+\omega}$ etc., from the point of view of the existence of countably compact group topologies.

Corollary 2.23. *For every ordinal $\sigma \geq 1$, it is consistent with ZFC and $\mathfrak{c} = \omega_1$ that every Abelian group of size $\aleph_\sigma$ satisfying conditions* **PS** *and* **CC** *admits a (hereditarily separable) countably compact group topology (without infinite compact subsets).*

Proof. "Make" κ bigger than $\aleph_\sigma$. Then, in $M_\kappa[\mathbb{G}]$, $2^{\mathfrak{c}} = \kappa$ will also be bigger than $\aleph_\sigma$. Now our corollary immediately follows from the conclusion of Theorem 2.7. □

Again, things become especially transparent in both torsion and torsion-free case.

Corollary 2.24. *For every ordinal $\sigma \geq 1$, it is consistent with ZFC plus $\mathfrak{c} = \omega_1$ that every bounded torsion Abelian group of size $\aleph_\sigma$ satisfying* **CC** *admits a (hereditarily separable) countably compact group topology (without infinite compact subsets).*

Proof. This follows from Corollary 2.23 because bounded torsion groups satisfy **PS** (see item (ii) of Lemma 2.4). □

Corollary 2.25. *For every ordinal $\sigma \geq 1$, it is consistent with ZFC plus $\mathfrak{c} = \omega_1$ that for every prime number p and each natural number $n \geq 1$ the group $\mathbb{Z}(p^n)^{(\aleph_\sigma)}$ admits a (hereditarily separable) countably compact group topology (without infinite compact subsets).*

Proof. This follows from Corollary 2.24 since the group $\mathbb{Z}(p^n)^{(\aleph_\sigma)}$ satisfies condition **CC** because $\mathfrak{c} = \aleph_1 \leq \aleph_\sigma$. □

Even a particular case of our last corollary, with $p = 2$ and $n = 1$, implies the main result of Tomita [44]: For every ordinal $\sigma \geq 1$, it is consistent with ZFC plus $\mathfrak{c} = \omega_1$ that the Boolean group $\mathbb{Z}(2)^{(\aleph_\sigma)}$ of size $\aleph_\sigma$ can be equipped with a countably compact group topology. It is also worth mentioning that the group topology constructed in [44] is not hereditarily separable and has non-trivial convergent sequences (because it contains a Σ-product of uncountably many compact metric groups, and it is easily seen that such a Σ-product is not separable and has an infinite compact metric subgroup).

Corollary 2.26. *For every ordinal $\sigma \geq 1$, it is consistent with ZFC that every torsion-free Abelian group of size $\aleph_\sigma$ admits a (hereditarily separable) countably compact group topology (without infinite compact subsets).*

Proof. "Make" κ bigger than $\aleph_\sigma$. Then, in $M_\kappa[\mathbb{G}]$, $2^{\mathfrak{c}} = \kappa$ will also be bigger than $\aleph_\sigma$. Since $\mathfrak{c} = \aleph_1 \leq \aleph_\sigma$, Corollary 2.13 applies. □

Our results on hereditary separable topologizations allow us to make a contribution to the celebrated "S-space problem" . Scattered examples of topological groups which are S-spaces are known in the literature [21, 15, 38, 28, 33, 34, 29, 37]. Our final three theorems describe completely which Abelian groups admit group topologies (with various compactness conditions) which make them into S-spaces.

Theorem 2.27. *In $M_\kappa[\mathbb{G}]$, the following are equivalent for an Abelian group G:*
(i) G admits a group topology that makes it into an S-space,
(ii) G admits a precompact group topology that makes it into an S-space,
(iii) $\mathfrak{c} \leq |G| \leq 2^{\mathfrak{c}}$.

Theorem 2.28. *In $M_\kappa[\mathbb{G}]$, the following are equivalent for an Abelian group G:*
(i) G admits a pseudocompact group topology that makes it into an S-space,
(ii) $\mathfrak{c} \leq |G| \leq 2^{\mathfrak{c}}$ and G satisfies both **PS** *and* **tCC**.

Theorem 2.29. *In $M_\kappa[\mathbb{G}]$, the following are equivalent for an Abelian group G:*
(i) G admits a countably compact group topology that makes it into an S-space,
(ii) $\mathfrak{c} \leq |G| \leq 2^{\mathfrak{c}}$ and G satisfies both **PS** *and* **CC**.

Since hereditarily separable (initially ω_1-)compact groups are metrizable, a (initially ω_1-)compact group cannot be an S-space.

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(D. Dikranjan)
Dipartimento di Matematica e Informatica
Università di Udine
Via delle Scienze 206, 33100 Udine
Italy
E-mail address: dikranja@dimi.uniud.it

(D. Shakhmatov)
Department of Mathematical Sciences, Faculty of Science
Ehime University, Matsuyama 790–8577
Japan
E-mail address: dmitri@dpc.ehime-u.ac.jp

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ON THE PROPERTY "SEP" OF PARTIAL ORDERINGS

SAKAÉ FUCHINO

ABSTRACT. *A partial ordering P has the property* SEP *(or* SEP(P)*), if, for a sufficiently large regular cardinal χ, the family of elementary submodels of $\mathcal{H}(\chi)$ of cardinality $\aleph_1$ with the property that $[M]^{\aleph_0} \cap M$ is cofinal subset (with respect to $\subseteq$) of $[M]^{\aleph_0}$ and $P \cap M \leq_\sigma P$, is cofinal (also with respect to $\subseteq$) in $[\mathcal{H}(\chi)]^{\aleph_1}$ (see* Proposition 2 *below). We describe, in section 1, against which historical background this notion came to be formulated. In section 2, we introduce the property SEP and some other related properties of partial orderings P such as the weak Freese-Nation property and $(\aleph_1, \aleph_0)$-ideal property and review some known results around these properties. In section 3 we give a sketch of a proof of the theorem asserting that the combinatorial principle* PRINC *introduced by S. Shelah does not imply* SEP *of $\langle \mathcal{P}(\omega), \subseteq \rangle$.*

The following is based on author's talk at General Topology Symposium 2002 held on 18–20, November 2002 at Kobe University, Japan. The property SEP to be introduced in section 2 will be the main subject of this article. But since this is going to appear in the proceedings of the general topology symposium, let us begin with a historical account explaining the connection to topology.

1. HISTORICAL BACKGROUND

Remember that a zero-dimensional Hausdorff space is also called a Boolean space — a topological space X is said to be zero-dimensional if closed and open (clopen) subsets of X constitute a topological base of X.

It is well-known that there are contravariant functors between the category of Boolean algebras and the category of Boolean spaces (Stone Duality Theorem). By the duality, a Boolean algebra B is related to the space $Ult(B)$ of all ultra-filters on B with a canonical topology and a Boolean space X is related to the Boolean algebra $Clop(B)$ of clopen subsets of X partially ordered by $\subseteq$ (see e.g. [16]).

One of the most important classes of Boolean spaces is that of the generalized Cantor spaces ${}^\kappa 2$ which are the product spaces of κ copies of the discrete space $2 = \{0, 1\}$ with their Boolean algebraic dual being free Boolean algebras $Fr(\kappa)$ with a free generator of size κ. A variety of classes of topological spaces which are more or less similar to the generalized Cantor spaces, such as dyadic spaces, Dugundji spaces and κ-metrizable spaces, have been studied extensively in the literature (see e.g. the reference of [17] and [12]).

These classes of topological spaces have natural counterparts in the category of Boolean algebras via Stone Duality Theorem (see [17], [12]):

Boolean space which is:	corresponding notion in Boolean algebra
${}^{\kappa}2$	$Fr(\kappa)$
dyadic space	subalgebra of a free Boolean algebra
Dugundji space	projective Boolean algebra
κ-metrizable space	Boolean algebra with Freese-Nation property (also called openly generated Boolean algebra)

The last line of the chart above is a result by Lutz Heindorf in [12] which can be formulated as follows:

Theorem 1.1. (L. Heindorf [12]) *A Boolean algebra B is a Boolean algebraic dual of a κ-metrizable space if and only if there is a mapping $f : B \to [B]^{<\aleph_0}$ such that*

($*$) *For any $a,\, b \in B$, $a \le b$, there is $c \in f(a) \cap f(b)$ such that $a \le c \le b$.*

A Boolean algebra with an f as above is said to have the Freese-Nation property after R. Freese and B. Nation who studied this property in connection with projective lattices in [2]. Soon after this result, an interesting weakening of the Freese-Nation property was formulated in [12] and [9]: a Boolean algebra B is said to have the weak Freese-Nation property (wFN) if there is a mapping $f : B \to [B]^{\aleph_0}$ with the property ($*$) as above. For a topological translation of the wFN see [12].

The following is an almost trivial but very useful characterization of Boolean algebras with the wFN:

Theorem 1.2. (S. Fuchino, S. Koppelberg and S. Shelah [9]) *A Boolean algebra B has the wFN if and only if, for any sufficiently large regular κ and for any $M \prec \langle \mathcal{H}(\chi), \in \rangle$ with $B \in M$ and $|\, M \,| = \aleph_1$, we have $B \cap M \le_\sigma B$.*

Here $\mathcal{H}(\chi)$ denotes the set consisting of all sets of hereditary of cardinality $< \chi$. For a partial ordering $\langle P, \le \rangle$ and its subordering Q, Q is said to be a σ-subordering of P (or $Q \le_\sigma P$) if and only if for any $p \in P$, $Q \restriction p = \{q \in Q \,:\, q \le p\}$ has a cofinal subset of size $\le \aleph_0$ and $Q \uparrow p = \{q \in Q \,:\, q \ge p\}$ has a coinitial subset of size $\le \aleph_0$. Note that, in case of a Boolean algebra B and its subalgebra A, it is enough to check that every ideal in A of the form $A \restriction b = \{a \in A \,:\, a \le b\}$ for some $b \in B$ is countably generated to see that A is a σ-subalgebra of B.

Let us say that a partial ordering P has the wFN (notation: WFN(P)) if P satisfies the property given in Theorem 1.2, i.e. if for any sufficiently large regular κ and for any $M \prec \langle \mathcal{H}(\chi), \in \rangle$ with $P \in M$ and $|\, M \,| = \aleph_1$, we have $P \cap M \le_\sigma P$.

An interesting point about the wFN is that also the Boolean algebra $\langle \mathcal{P}(\omega), \subseteq \rangle$ can have this property†. Under CH this is trivially so but also in a model of ZFC obtained by adding Cohen reals e.g. to a model of $V = L$ (see [9] and [10]).

In [8], it is shown that under the assumption‡ of the wFN of $\langle \mathcal{P}(\omega), \subseteq \rangle$, we can prove many of the combinatorial statements known to hold in Cohen models.

2. SEP and some other weakenings of the weak Freese-Nation property

A. Dow and K.P. Hart defined the following weakening of the property WFN(P) of a partial ordering P in [1]: A partial ordering P has the $(\aleph_1, \aleph_0)$-Ideal Property (abbreviation: IDP(P)) if the following holds:

†In contrast, $\langle \mathcal{P}(\omega), \subseteq \rangle$ never has the Freese-Nation property.

‡This assumption is also denoted as "WFN".

IDP(P): *For any sufficiently large regular κ and for any $M \prec \langle \mathcal{H}(\chi), \in \rangle$ with $P \in M$ and $|M| = \aleph_1$ such that $[M]^{\aleph_0} \cap M$ is cofinal in $[M]^{\aleph_0}$, we have $P \cap M \leq_\sigma P$.*

The question if WFN(P) is equivalent with IDP(P) is a very delicate one:

Theorem 2.1. (S. Fuchino and L. Soukup [10]) *For partial ordering P of cardinality $< \aleph_\omega$,* WFN(P) *if and only if* IDP(P). *If a very weak form of square principle holds for cardinals of countable cofinality then* WFN(P) *and* IDP(P) *are equivalent for arbitrary partial ordering P.*

On the other hand, as is proved in [10], there is a partial ordering P with $\neg$WFN(P) and IDP(P) under $(\aleph_\omega)^{\aleph_0} = \aleph_{\omega+1}$ and Chang's conjecture $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$.

It is shown in [1](independently from [8]) that most of the results obtained in [8] under the wFN of $\mathcal{P}(\omega)$ can be already proved under the weaker assumption of IDP of $\mathcal{P}(\omega)$.

In [13], I. Juhász and K. Kunen introduced a property which they called SEP. For a Boolean algebra B let SEP(B) be the following assertion:

SEP(B): *For every sufficiently large regular cardinal χ, the set of those $M \prec \mathcal{H}(\chi)$ satisfying the following conditions* (0)$\sim$(2) *is cofinal in $[\mathcal{H}(\chi)]^{\aleph_1}$:*

(0) *$B \in M$ and $|M| = \aleph_1$;*
(1) *$[M]^{\aleph_0} \cap M$ is cofinal in $[M]^{\aleph_0}$;*
(2) *$B \cap M \leq_{sep} B$.*

Here, for a Boolean algebra A and its subalgebra B, $B \leq_{sep} A$ if and only if for all $a \in A$ and $K \in [B \restriction a]^{\aleph_1}$, there is $b \in B \restriction a$ such that $|K \cap B \restriction b| = \aleph_1$ where $B \restriction a$ denotes as before the ideal $\{d \in B : d \leq a\}$.

SEP has a characterization which put it in line with wFN and IDP. This follows from the next lemma. (S. Fuchino and S. Geschke [6]) Suppose that A is a Boolean algebra and B its subalgebra. Then

(1) $B \leq_\sigma A$ implies $B \leq_{sep} A$.
(2) If $|B| \leq \aleph_1$ then $B \leq_\sigma A$ implies $B \leq_{sep} A$

Proof. (1): Suppose that $B \leq_\sigma A$. For $a \in A$ and $K \in [B \restriction a]^{\aleph_1}$. Let $\{b_n : n \in \omega\}$ be a cofinal subset of $B \restriction a$. Then $K = \bigcup_{n\in\omega}(K \cap B \restriction b_n)$. Hence one of $K \cap B \restriction b_n$, $n \in \omega$ must be uncountable.

(2): Let $B = \{b_\alpha : \alpha < \omega_1\}$. Assume that $B \leq_{sep} A$ but B is not a σ-subalgebra of A. Then there is an $a \in A$ such that $B \restriction a$ is not countably generated. Let $c_\alpha \in A \restriction b$ be taken inductively so that c_α is not in the ideal in B generated by

$$G_\alpha = \{c_\beta : \beta < \alpha\} \cup \{b_\beta : \beta < \alpha,\, b_\beta \leq a\}.$$

This is possible since each $G_\alpha \subseteq B \restriction a$ is countable. Let $K = \{c_\alpha : \alpha < \omega_1\}$. By assumption there is some $\beta_0 < \omega_1$ such that $b_{\beta_0} \leq a$ and $K \cap B \restriction b_{\beta_0}$ is uncountable. Then there is some $\beta_0 < \alpha < \omega_1$ such that $c_\alpha \in K \cap B \restriction b_{\beta_0}$. But, since $b_{\beta_0} < a$, $b_{\beta_0} \in G_\alpha$ and so $c_\alpha \not\leq b_{\beta_0}$. This is a contradiction. ❑ (Lemma 0)

(S. Fuchino and S. Geschke [6]) For a Boolean algebra B, SEP(B) if and only if the following holds:

For every sufficiently large regular cardinal χ, the set of those $M \prec \mathcal{H}(\chi)$ satisfying the following conditions (0)∼(2) *is cofinal in* $[\mathcal{H}(\chi)]^{\aleph_1}$:

(0) $B \in M$ *and* $|\,M\,| = \aleph_1$*;*
(1) $[M]^{\aleph_0} \cap M$ *is cofinal in* $[M]^{\aleph_0}$*;*
(2') $B \cap M \leq_\sigma B$*.*

Proof. By Lemma 2. ❑ (Proposition 0)

Note that the characterization of SEP is applicable for partial orderings as well. Hence we shall say SEP(P) for a partial ordering P if, for every sufficiently large regular cardinal χ, the set of those $M \prec \mathcal{H}(\chi)$ satisfying the following conditions (0)∼(2') is cofinal in $[\mathcal{H}(\chi)]^{\aleph_1}$: (0) $P \in M$ and $|\,M\,| = \aleph_1$; (1) $[M]^{\aleph_0} \cap M$ is cofinal in $[M]^{\aleph_0}$; (2') $P \cap M \leq_\sigma P$.

Clearly we have

$$\mathrm{WFN}(P) \;\Rightarrow\; \mathrm{IDP}(P) \;\Rightarrow\; \mathrm{SEP}(P)$$

for all partial ordering P. In [6] it is proved (in ZFC without any additional assumptions) that there exists a Boolean algebra B such that ¬IDP(B) and SEP(B).

3. Princ does not imply SEP

Let us denote by WFN, IDP and SEP the combinatorial statements WFN($\mathcal{P}(\omega)$), IDP($\mathcal{P}(\omega)$) and SEP($\mathcal{P}(\omega)$) respectively where $\mathcal{P}(\omega)$ is seen here as before as the partial ordering $\langle \mathcal{P}(\omega), \subseteq \rangle$. It is shown in [7] that WFN $\not\Leftarrow$ IDP is consistent with ZFC modulo some quite large cardinal. The consistency of IDP $\not\Leftarrow$ SEP (without any large cardinal) is proved in [6]. Thus we have:

$$\mathrm{WFN} \;\underset{\not\Leftarrow}{\Rightarrow}\; \mathrm{IDP} \;\underset{\not\Leftarrow}{\Rightarrow}\; \mathrm{SEP}$$

SEP is still strong enough to drive most of the results proved in [8] under the assumption of WFN. Most of the proofs in [8] can be easily modified to a proof under SEP. With one exception: The proof of $\mathfrak{a} = \aleph_1$ under WFN in [8] works without any problem under IDP but it seems that the proof cannot be recast for the proof under SEP. What we have right now is the following slightly weaker result:

Theorem 3.1. (S. Fuchino and S. Geschke [6]) *Assume* $\square_{\omega_1}$*. Then* SEP *implies* $\mathfrak{a} = \aleph_1$*.*

There are some other combinatorial principles which are also related to these principles. One of them is introduced by S. Shelah and called Princ:

Princ: *For every sufficiently large regular cardinal χ, the set of those $M \prec \mathcal{H}(\chi)$ satisfying the following conditions* (I)∼(III) *is cofinal in* $[\mathcal{H}(\chi)]^{\aleph_1}$:

(I) $|\,M\,| = \aleph_1$*;*
(II) $\omega_2 \cap M \in \omega_2$*;*
(III) *For all $a \in \mathcal{P}(\omega)$ there is $X \in [\mathcal{P}(\omega)]^{\aleph_1} \cap M$ such that $X \cap \mathcal{P}(a)$ is cofinal in $\mathcal{P}(a) \cap M$.*

SEP implies Princ. *Proof.* Assume SEP and suppose that χ is sufficiently large and $M \prec \mathcal{H}(\chi)$ is such that $|\,M\,| = \aleph_1$, $[M]^{\aleph_0} \cap M$ is cofinal in $[M]^{\aleph_0}$ and $\mathcal{P}(\omega) \cap M \leq_\sigma \mathcal{P}(\omega)$.

Note that, by assumption, there are cofinally may such M's in $[\mathcal{H}(\chi)]^{\aleph_1}$. Hence it is enough to show that M as above satisfies (I), (II) and (III) in the definition of Princ.

$M \models$ (I) is clear. For (II), note that $\omega_1 \subseteq M$. Suppose $\alpha \in \omega_2 \cap M$. Then, by elementarity, there is $f \in M$ such that $f : \omega_1 \to \alpha$ and f is surjective. It follows that $\alpha = f''\omega_1 \subseteq M$. Hence $\omega_2 \cap M$ is an initial segment of ω_2 of cardinality $\aleph_1$ and thus an element of ω_2.

To show $M \models$ (III), let $a \in \mathcal{P}(\omega)$. Then there is a countable $X' \subseteq \mathcal{P}(a) \cap M$ such that X' is cofinal in $\mathcal{P}(a) \cap M$. Let $X'' \in [\mathcal{P}(\omega)]^{\aleph_0} \cap M$ be such that $X' \subseteq X''$. By elementarity, there is some $X \in [\mathcal{P}(\omega)]^{\aleph_1} \cap M$ with $X'' \subseteq X$. This X is clearly as in (III) for our a. ❑ (Lemma 0)

To close this section, we shall give a sketch of the proof the the converse of Lemma 0does not hold. The following Theorem is due to Stefan Geschke:

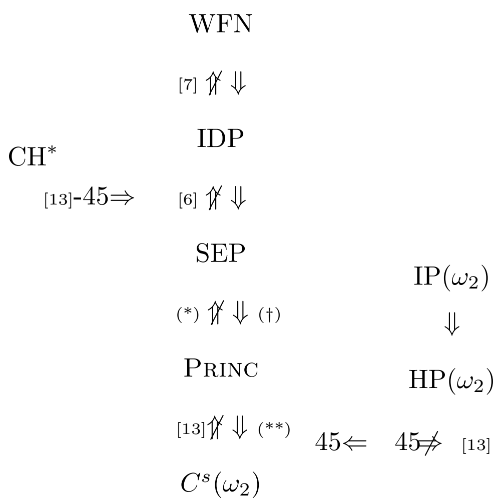
Theorem 3.2. *Suppose that $V \models$ CH, Q is a partial ordering satisfying the c.c.c. and there is a partial ordering P of cardinality $\leq \aleph_1$ such that Q is a finite support product of copies of P. Then $\Vdash_Q$ "*PRINC*".*

Theorem 3.3. $\neg$SEP + PRINC *is consistent with* ZFC.

Proof. Start from a model V of CH. In V, let P be the partial ordering for adding a single random real and let Q be a finite support product of $\kappa > \aleph_1$ copies of P. Then $\Vdash_Q$ "$\neg$SEP" by Lemma 3.1.6 in [11] and Theorem 8.1 in [6]. On the other hand $\Vdash_Q$ "PRINC" by Theorem 3.2. ❑ (Theorem 0)

4. Some more principles

The following diagram summarizes known implications among combinatorial principles discussed in the previous sections together with some other combinatorial principles from [14], [15] and [5]:



(†) By Lemma 3.
(*) By Theorem 3.3.
(**) By S. Shelah in an unpublished note.

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Department of Natural Science and Mathematics, Chubu University. Kasugai Aichi 487-8501 JAPAN

E-mail address: `fuchino@isc.chubu.ac.jp`

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SELECTIONS AND SANDWICH-LIKE PROPERTIES VIA SEMI-CONTINUOUS BANACH-VALUED FUNCTIONS

VALENTIN GUTEV, HARUTO OHTA, AND KAORI YAMAZAKI

1. Introduction

Since the paper will be published in the J. Math. Soc. Math. soon, we omit all of the proofs and some lemmas. Throughout this paper, by a space we mean a non-empty T_1-space. Our investigation was motivated by the following two theorems; the former was proved by Katětov [14, 15] and Tong [29], and the latter was proved by Kandô [13] and Nedev [23]:

Theorem 1.1 (Katětov-Tong's insertion theorem)**.** *A space X is normal if and only if for every two functions $g, h : X \to \mathbb{R}$ such that g is upper semi-continuous, h is lower semi-continuous and $g \leq h$, there exists a continuous function $f : X \to \mathbb{R}$ such that $g \leq f \leq h$.*

For a Banach space Y, let $\mathcal{F}_c(Y)$ (resp., $\mathcal{C}_c(Y)$) denote the set of all non-empty closed (resp., non-empty compact) convex sets in Y. A map $f : X \to Y$ is called a *selection* of a mapping $\phi : X \to \mathcal{F}_c(Y)$ if $f(x) \in \phi(x)$ for every $x \in X$.

Theorem 1.2 (Kandô-Nedev's selection theorem)**.** *Let λ be an infinite cardinal. Then, the following conditions on a space X are equivalent*:

1. *Every point-finite open cover $\mathcal{U}$ of X, with $|\mathcal{U}| \leq \lambda$, is normal.*
2. *For every Banach space Y, with $w(Y) \leq \lambda$, every lower semi-continuous mapping $\phi : X \to \mathcal{C}_c(Y)$ admits a continuous selection.*
3. *Every lower semi-continuous mapping $\phi : X \to \mathcal{C}_c(\ell_1(\lambda))$ admits a continuous selection.*

Theorem 1.2 can be regarded as an essential part of Michael's selection theorem [19, Theorem 3.2′] (see, also, [2]) asserting that a space X is λ-collectionwise normal if and only if X satisfies the condition (2) with $\mathcal{C}_c(Y)$ replaced by $\mathcal{C}_c(Y) \cup \{Y\}$.

For a space Y, let $C_0(Y)$ denote the Banach space of all real-valued continuous functions s on Y such that for each $\varepsilon > 0$ the set $\{y \in Y : |s(y)| \geq \varepsilon\}$ is compact, where the linear operations are defined pointwise and $\|s\| = \sup_{y \in Y} |s(y)|$ for each $s \in C_0(Y)$. In particular, we use $c_0(\lambda)$ to denote the space $C_0(Y)$, where Y is the discrete space of cardinality λ, i.e. $c_0(\lambda)$ is the Banach space consisting of all points $s \in \mathbb{R}^\lambda$ such that the set $\{\alpha < \lambda : |s(\alpha)| \geq \varepsilon\}$ is finite for each $\varepsilon > 0$.

In this paper, we introduce lower and upper semi-continuity of a map to $C_0(Y)$. We prove that if the space $\mathbb{R}$ in Theorem 1.1 is replaced by $c_0(\lambda)$, then the resulting statement is equivalent to the conditions listed in Theorem 1.2, see Theorem 3.1. Thus, insertions and selections are connected via the space $c_0(\lambda)$. As a result, we obtain several sandwich-like analogues to selection theorems as well as selection theorems corresponding to sandwich-like properties, see Section 4.

For set-valued mappings φ and ψ defined on a space X, we say that φ is a *set-valued selection* of ψ, or ψ is an *expansion* of φ, if $\varphi(x) \subseteq \psi(x)$ for each $x \in X$. Let $\mathcal{C}(Y)$ denote the set of all non-empty compact sets in a space Y. In [23] Nedev has characterized several paracompact-like properties by the existence of set-valued selections of $\mathcal{C}(Y)$-valued mappings for completely metrizable spaces Y. In contrast to this, we characterize expandability and almost expandability in the sense of [16, 27] by insertion of $c_0(\lambda)$-valued maps, and by the existence of expansions of $\mathcal{C}(Y)$-valued mappings for completely metrizable spaces Y, see Section 5.

We often consider two kinds of maps in the same statement, i.e., a single-valued map to a space Y and a set-valued map to a hyperspace of Y. To distinguish them, we use the term *map* for the former one and the term *mapping* for the latter one. As usual, a cardinal is identified with the initial ordinal and an ordinal is the set of all smaller ordinals. The cardinality of a set A is denoted by $|A|$. Let ω denote the first infinite cardinal and $\mathbb{N}$ the set of non-negative integers. Other terms and notation will be used as in [8].

2. Semi-continuous $C_0(Y)$-valued functions and compact sets

In this section, X and Y denote arbitrary spaces and λ stands for a cardinal. For a real-valued function $f : X \to \mathbb{R}$ and $r \in \mathbb{R}$, let $L(f, r) = \{x \in X : f(x) > r\}$ and $U(f, r) = \{x \in X : f(x) < r\}$. Recall that a function $f : X \to \mathbb{R}$ is *lower* (resp., *upper*) *semi-continuous* if $L(f, r)$ (resp., $U(f, r)$) is open in X for each $r \in \mathbb{R}$. Now, we extend these notions to $C_0(Y)$-valued maps as follows:

Definition 2.1. A map $f : X \to C_0(Y)$ is *lower* (resp., *upper*) *semi-continuous* if for every $x \in X$ and every $\varepsilon > 0$, there is a neighbourhood G of x in X such that if $x' \in G$, then $f(x')(y) > f(x)(y) - \varepsilon$ (resp., $f(x')(y) < f(x)(y) + \varepsilon$) for each $y \in Y$.

With every map $f : X \to C_0(Y)$ we associate another one $-f : X \to C_0(Y)$ defined by $(-f)(x)(y) = -f(x)(y)$ for each $x \in X$ and each $y \in Y$. The first lemma is a direct consequence of the definition.

Lemma 2.2. *A map $f : X \to C_0(Y)$ is continuous if and only if it is both lower and upper semi-continuous. A map $f : X \to C_0(Y)$ is lower semi-continuous if and only if the map $-f$ is upper semi-continuous.*

The following three lemmas concern only the case of $c_0(\lambda)$. For each $\alpha < \lambda$, let $\pi_\alpha : \mathbb{R}^\lambda \to \mathbb{R}$ denote the α-th projection, i.e. $\pi_\alpha(s) = s(\alpha)$ for $s \in \mathbb{R}^\lambda$.

Lemma 2.3. *For a map $f : X \to c_0(\lambda)$, the following are valid:*

(1) *f is lower semi-continuous if and only if $\pi_\alpha \circ f$ is lower semi-continuous for each $\alpha < \lambda$, and $\{U(\pi_\alpha \circ f, -\varepsilon) : \alpha < \lambda\}$ is locally finite in X for each $\varepsilon > 0$.*

(2) *f is upper semi-continuous if and only if $\pi_\alpha \circ f$ is upper semi-continuous for each $\alpha < \lambda$, and $\{L(\pi_\alpha \circ f, \varepsilon) : \alpha < \lambda\}$ is locally finite in X for each $\varepsilon > 0$.*

Lemma 2.4. *Let $f : X \to \mathbb{R}^\lambda$ be a map. Then, $f[X] \subseteq c_0(\lambda)$ if and only if both $\{L(\pi_\alpha \circ f, \varepsilon) : \alpha < \lambda\}$ and $\{U(\pi_\alpha \circ f, -\varepsilon) : \alpha < \lambda\}$ are point-finite in X for each $\varepsilon > 0$.*

For $s \in C_0(Y)$ and $\varepsilon > 0$, let $B(s,\varepsilon) = \{t \in C_0(Y) : \|s-t\| < \varepsilon\}$. For $s,t \in C_0(Y)$, we write $s \le t$ if $s(y) \le t(y)$ for each $y \in Y$. Further, if $s \le t$, then we define $[s,t] = \{u \in \mathbb{R}^Y : s \le u \le t\}$. Obviously, $[s,t]$ is a closed convex subset of $C_0(Y)$. In the case of $c_0(\lambda)$, we have a stronger result:

Lemma 2.5. *For every $s,t \in c_0(\lambda)$, with $s \le t$, the subspace topology σ on $[s,t]$ coincides with the subspace topology induced from the product topology τ on $\mathbb{R}^\lambda$. Hence, in particular, $[s,t]$ is a compact convex subset of $c_0(\lambda)$.*

We now recall the definitions of upper and lower semi-continuity of set-valued mappings. Let $\phi : X \to \mathcal{S}$ be a set-valued mapping, where $\mathcal{S}$ is a family of non-empty subsets of a space Y. For a subset $U \subseteq Y$, let $\phi^{-1}[U] = \{x \in X : \phi(x) \cap U \neq \emptyset\}$ and $\phi^{\#}[U] = \{x \in X : \phi(x) \subseteq U\}$. The mapping $\phi : X \to \mathcal{S}$ is called *lower* (resp., *upper*) *semi-continuous* if $\phi^{-1}[U]$ (resp., $\phi^{\#}[U]$) is open in X for every open set U in Y. Also, ϕ is called *continuous* if it is both lower and upper semi-continuous.

For maps $g,h : X \to C_0(Y)$, we shall write $g \le h$ if $g(x) \le h(x)$ for every $x \in X$. With every two such maps we associate a set-valued mapping $[g,h] : X \to \mathcal{F}_c(C_0(Y))$ defined by $[g,h](x) = [g(x),h(x)]$ for $x \in X$. Also, we associate two mappings $[g,+\infty)$ and $(-\infty,h]$ from X to $\mathcal{F}_c(C_0(Y))$ by $[g,+\infty)(x) = \{s \in C_0(Y) : s \ge g(x)\}$ and $(-\infty,h](x) = \{s \in C_0(Y) : s \le h(x)\}$ for $x \in X$, respectively. Finally, for $S \subseteq C_0(Y)$ and $\varepsilon > 0$, let $B(S,\varepsilon)$ denote the ε-neighbourhood of S in $C_0(Y)$, i.e. $B(S,\varepsilon) = \bigcup_{s\in S} B(s,\varepsilon)$.

Lemma 2.6. *Let $g,h : X \to C_0(Y)$ be maps such that $g \le h$.*

(1) *If g is upper semi-continuous, then $[g,+\infty)$ is lower semi-continuous.*
(2) *If h is lower semi-continuous, then $(-\infty,h]$ is lower semi-continuous.*
(3) *If g is upper semi-continuous and h is lower semi-continuous, then the mapping $[g,h]$ is lower semi-continuous.*
(4) *If g is lower semi-continuous, h is upper semi-continuous and Y is discrete, then the mapping $[g,h]$ is upper semi-continuous.*

For a non-empty bounded set $K \subseteq C_0(Y)$, we define points $\sup K$ and $\inf K$ of $\mathbb{R}^Y$ by $(\sup K)(y) = \sup\{s(y) : s \in K\}$ and $(\inf K)(y) = \inf\{s(y) : s \in K\}$, respectively, for each $y \in Y$.

Lemma 2.7. *If K is a non-empty compact set in $C_0(Y)$, then $\sup K \in C_0(Y)$ and $\inf K \in C_0(Y)$. Hence, $K \subseteq [\inf K, \sup K]$.*

For a mapping $\phi : X \to \mathcal{C}(C_0(Y))$, we define single-valued maps $\sup\phi : X \to C_0(Y)$ and $\inf\phi : X \to C_0(Y)$ by $(\sup\phi)(x) = \sup\phi(x)$ and $(\inf\phi)(x) = \inf\phi(x)$, respectively, for each $x \in X$.

Lemma 2.8. *Let $\phi : X \to \mathcal{C}(C_0(Y))$ be a mapping.*

(1) *If ϕ is lower semi-continuous, then $\sup\phi$ is lower semi-continuous and $\inf\phi$ is upper semi-continuous.*
(2) *If ϕ is upper semi-continuous, then $\sup\phi$ is upper semi-continuous and $\inf\phi$ is lower semi-continuous.*

3. Extension of Theorem 1.2

For two families $\mathcal{F}$ and $\mathcal{G}$ of subsets of a space X, we call $\mathcal{G}$ an *expansion* of $\mathcal{F}$ if there exists a bijection $G : \mathcal{F} \to \mathcal{G}$ such that $F \subseteq G(F)$ for each $F \in \mathcal{F}$. An *open expansion* is an expansion consisting of open sets. For real-valued functions f_α, $\alpha < \lambda$, on a space X, let $\triangle_{\alpha<\lambda} f_\alpha$ denote the map $f : X \to \mathbb{R}^\lambda$ such that $\pi_\alpha \circ f = f_\alpha$ for each $\alpha < \lambda$.

In this section, we find a natural relationship between insertions and selections by proving the following theorem which extends Theorem 1.2. The equivalence of (1) and (2) is due to Kandô [13] and Nedev [23] as was stated in the introduction.

Theorem 3.1. *For an infinite cardinal λ, the following conditions on a space X are equivalent*:

(1) *Every point-finite open cover $\mathcal{U}$ of X, with $|\mathcal{U}| \leq \lambda$, is normal.*
(2) *For every Banach space Y, with $w(Y) \leq \lambda$, every lower semi-continuous mapping $\varphi : X \to \mathcal{C}_c(Y)$ admits a continuous selection.*
(3) *Every lower semi-continuous mapping $\varphi : X \to \mathcal{C}_c(c_0(\lambda))$ admits a continuous selection.*
(4) *For every two maps $g, h : X \to c_0(\lambda)$ such that g is upper semi-continuous, h is lower semi-continuous and $g \leq h$, there exists a continuous map $f : X \to c_0(\lambda)$ such that $g \leq f \leq h$.*
(5) *For every two maps $g, h : X \to c_0(\lambda)$ such that g is upper semi-continuous, h is lower semi-continuous and $g \leq h$, there exist a lower semi-continuous map $f_\ell : X \to c_0(\lambda)$ and an upper semi-continuous map $f_u : X \to c_0(\lambda)$ such that $g \leq f_\ell \leq f_u \leq h$.*
(6) *X is normal, and every locally finite family $\mathcal{F}$ of closed sets in X, with $|\mathcal{F}| \leq \lambda$, has a locally finite open expansion provided it has a point-finite open expansion.*
(7) *Every discrete family $\mathcal{F}$ of closed sets in X, with $|\mathcal{F}| \leq \lambda$, has a disjoint open expansion provided it has a point-finite open expansion.*

Remark 3.2. The following conditions (8) and (9) are also equivalent to the conditions listed in Theorem 3.1. For two mappings $\varphi, \psi : X \to \mathcal{C}(Y)$, we write $\varphi \subseteq \psi$ if $\varphi(x) \subseteq \psi(x)$ for each $x \in X$.

(8) For every metrizable space Y, with $w(Y) \leq \lambda$, and every lower semi-continuous mapping $\phi : X \to \mathcal{C}(Y)$, there exist a lower semi-continuous mapping $\varphi : X \to \mathcal{C}(Y)$ and an upper semi-continuous mapping $\psi : X \to \mathcal{C}(Y)$ such that $\varphi \subseteq \psi \subseteq \phi$.
(9) There exist a space Y and a disjoint family $\mathcal{G}$ of non-empty open sets in Y, with $|\mathcal{G}| = \lambda$, such that for every lower semi-continuous mapping $\phi : X \to \mathcal{C}(Y)$, there exists an upper semi-continuous mapping $\psi : X \to \mathcal{C}(Y)$ such that $\psi \subseteq \phi$.

The equivalence of (1) and (8) was proved by Nedev in [23, Theorem 3], while (8) $\Rightarrow$ (9) is obvious. To show that (9) $\Rightarrow$ (7), let $\mathcal{F}$ be a discrete family of closed sets in X, with $|\mathcal{F}| \leq \lambda$, and $\mathcal{U} = \{U(F) : F \in \mathcal{F}\}$ be a point-finite open expansion of $\mathcal{F}$. We may assume that $\mathcal{U}$ covers X and $U(F) \cap F' = \emptyset$ whenever $F \neq F'$. On the other hand, there exists a disjoint family $\mathcal{G} = \{G(F) : F \in \mathcal{F}\}$ of non-empty open sets in Y. Fix a point $y_F \in G(F)$ for each $F \in \mathcal{F}$ and define $\phi : X \to \mathcal{C}(Y)$ by $\phi(x) = \{y_F : x \in U(F),\ F \in \mathcal{F}\}$ for $x \in X$. Then, ϕ is lower semi-continuous because $\mathcal{U}$ is an open cover of X. Hence, by (9), there exists an upper semi-continuous mapping $\psi : X \to \mathcal{C}(Y)$ such that $\psi \subseteq \phi$. Let $V(F) = \psi^{\#}[G(F)]$ for each $F \in \mathcal{F}$. Then $\{V(F) : F \in \mathcal{F}\}$ is a disjoint open expansion of $\mathcal{F}$.

Let λ-$\mathcal{PN}$ be the class of all spaces satisfying one of (and hence, all of) the conditions listed in Theorem 3.1. Define the class $\mathcal{PN}$ by $X \in \mathcal{PN}$ if and only if $X \in \lambda$-$\mathcal{PN}$ for every cardinal λ. Then, $\mathcal{PN}$ is included in the class $\mathcal{N}$ of all normal spaces and contains the class $\mathcal{CN}$ of all collectionwise normal spaces, i.e. $\mathcal{CN} \subseteq \mathcal{PN} \subseteq \mathcal{N}$. Michael [18] has shown that both inclusions are proper by giving the examples which we now sketch below:

The example showing that $\mathcal{PN} \neq \mathcal{CN}$ is the standard Bing's example (cf. [8, Example 5.1.23]). The product space $X = D^{2^{\mathfrak{c}}}$ of the discrete space $D = \{0, 1\}$ contains a discrete subspace M, with $|M| = \mathfrak{c}$. Bing's space Z is obtained from the space X by making all points of $X \backslash M$ isolated. It is known that $Z \in \mathcal{N} \backslash \mathcal{CN}$. Notice that every point-finite family of non-empty open sets in X is at most countable; this follows from the fact that the Šanin number of X is countable (cf. [8, 2.7.11, p. 116]). Hence, it follows that $Z \in \mathcal{PN}$. Next, consider the subspace $Y = M \cup D$ of Z, where $D = \{x \in X : \{\alpha < 2^{\mathfrak{c}} : x(\alpha) \neq 0\}$ is finite$\}$. Michael [18] has shown that the space Y is normal metacompact but not paracompact. Hence, $Y \in \mathcal{N} \setminus \mathcal{PN}$ because every metacompact space in $\mathcal{PN}$ must be paracompact.

Since the space $Y = M \cup D$ is closed in Bing's space Z, the example above also shows that the class $\mathcal{PN}$ is not closed under taking closed subspaces unlike $\mathcal{N}$ and $\mathcal{CN}$. From this fact, it is natural to ask whether a space X is in $\mathcal{CN}$ if every closed subspace of X is in $\mathcal{PN}$. Now, we show that the answer is negative if there exists a Q-set. To this end, let us recall that a subset A of the real line $\mathbb{R}$ is called a *Q-set* if A is uncountable and every subset of A is a G_δ-set in A with respect to the subspace topology on A inherited from the usual topology on $\mathbb{R}$. It is known that every uncountable subset $A \subseteq \mathbb{R}$, with $|A| < \mathfrak{c}$, is a Q-set under assuming Martin's axiom and the negation of the continuum hypothesis (see [20] for details).

Example 3.3. If there exists a Q-set in $\mathbb{R}$, then there exists a perfectly normal space X such that every subspace is in $\mathcal{PN}$ but $X \notin \mathcal{CN}$.

Problem 3.4. Does there exist an example in ZFC of a space $X \notin \mathcal{CN}$ such that every closed subspace of X is in $\mathcal{PN}$?

4. Sandwich-like characterizations of paracompact-like properties

A space X is called *λ-collectionwise normal* if every discrete family $\mathcal{F}$ of closed sets in X, with $|\mathcal{F}| \leq \lambda$, has a discrete open expansion. In what follows, for a Banach space Y, we put $\mathcal{C}'_c(Y) = \mathcal{C}_c(Y) \cup \{Y\}$.

Our first result is an insertion-like theorem which characterizes λ-collectionwise normality.

Theorem 4.1. *Let λ be an infinite cardinal. For a space X the following conditions are equivalent*:

(1) *X is λ-collectionwise normal.*
(2) *For every Banach space Y, with $w(Y) \leq \lambda$, every lower semi-continuous mapping $\varphi : X \to \mathcal{C}'_c(Y)$ has a continuous selection.*
(3) *Every lower semi-continuous mapping $\phi : X \to \mathcal{C}'_c(c_0(\lambda))$ has a continuous selection.*
(4) *For every closed subspace A of X and for every two maps $g, h : A \to c_0(\lambda)$ such that g is upper semi-continuous, h is lower semi-continuous and $g \leq h$, there exists a continuous map $f : X \to c_0(\lambda)$ such that $g \leq f|_A \leq h$.*

Our next result is a characterization of countably paracompact and λ-collectionwise normal spaces. To state our characterization of countably paracompact and λ-collectionwise normal spaces, we need also some terminology about Banach spaces.

Let Y be a space and let $e : Y \to \mathbb{R}^\lambda$ be a map. Then we define $e_\alpha = \pi_\alpha \circ e$, where $\pi_\alpha : \mathbb{R}^\lambda \to \mathbb{R}$ is the projection to the α-th factor of $\mathbb{R}^\lambda$, for each $\alpha < \lambda$. Thus, we have $e = \triangle\{e_\alpha : \alpha < \lambda\}$.

Suppose that Y is a Banach space. Let us recall that a sequence $\{\mathbf{e}_n \in Y : n < \omega\}$ is a *Schauder basis* for Y if any point $y \in Y$ has an unique representation $y = \sum_{n<\omega} y_n\mathbf{e}_n$ for some scalars (i.e., coordinates) $y_n \in \mathbb{R}$, $n < \omega$. Here, $y = \sum_{n<\omega} y_n\mathbf{e}_n$ means that $\lim_{n\to\infty} \left\| y - \sum_{k\le n} y_k\mathbf{e}_k \right\| = 0$, where $\|.\|$ is the norm of Y.

Note that any Schauder basis $\{\mathbf{e}_n \in Y : n < \omega\}$ for a Banach space Y defines a natural linear continuous injection $e : Y \to \mathbb{R}^\omega$, see [3, Exercise III.14.10] and [26, Theorem 3.1]. Namely, one may define $e : Y \to \mathbb{R}^\omega$ by $e_n(y) = y_n$, $n < \omega$, where $y = \sum_{n<\omega} y_n\mathbf{e}_n \in Y$. It should be mentioned that, with respect to this map $e = \triangle\{e_n : e < \omega\}$, we have $e_n(\mathbf{e}_n) = 1$ and $e_m(\mathbf{e}_n) = 0$ for $m \neq n$. Motivated by this, we shall say that a map $e : Y \to \mathbb{R}^\lambda$ is a *generalized Schauder basis* for a Banach space Y if it is a continuous linear injection such that, whenever $y \in Y$ and $\alpha < \lambda$, there is a point $y_\alpha \in Y$, with $e_\beta(y_\alpha) = e_\beta(y)$ if $\beta = \alpha$ and $e_\beta(y_\alpha) = 0$ otherwise. Clearly, the natural linear injection $e : Y \to \mathbb{R}^\omega$ determined by a Schauder basis for Y is a generalized Schauder basis but the converse does not hold. For instance, consider the Banach space ℓ^∞ of bounded sequences. Then the natural injection $e : \ell^\infty \to \mathbb{R}^\omega$ is a generalized Schauder basis but the space ℓ^∞ does not have a Schauder one since it is not separable.

The generalized Schauder basises will be used in the following special situation.

Definition 4.2. We shall say that a generalized Schauder basis $e : Y \to \mathbb{R}^\lambda$ for a Banach space Y is a $c_0(\lambda)$*-basis* for Y if $e[Y] \subset c_0(\lambda)$ and it is continuous as a map from Y to $c_0(\lambda)$. Also, we shall say that Y is a *generalized* $c_0(\lambda)$*-space* if it is a Banach space, with $w(Y) \le \lambda$, which has a $c_0(\lambda)$-basis.

Note that $c_0(\lambda)$ is a generalized $c_0(\lambda)$-space. Also, every Euclidean space is a generalized $c_0(\lambda)$-space for every infinite cardinal λ. Finally, the Banach spaces $\ell_p(\lambda)$, for $p \ge 1$, are another important example of generalized $c_0(\lambda)$-spaces.In

what follows, for a convex set K of a Banach space Y, we consider an *weak convex interior* $\mathrm{wci}(K)$ of K defined by

$$\mathrm{wci}(K) = \{x \in K : x = \delta x_1 + (1-\delta)x_2 \text{ for some } x_1, x_2 \in K \setminus \{x\} \text{ and } 0 < \delta < 1\}.$$

Also, for $s, t \in \mathbb{R}^\lambda$, we shall write $s < t$ if $s \le t$ and $s(\alpha) < t(\alpha)$ for some $\alpha < \lambda$. Finally, for maps $g, h : X \to \mathbb{R}^\lambda$, we write $g < h$ if $g(x) < h(x)$ for every $x \in X$.

Theorem 4.3. *Let λ be an infinite cardinal. For a space X the following conditions are equivalent*:

1. *X is countably paracompact and λ-collectionwise normal.*
2. *Whenever Y is a generalized $c_0(\lambda)$-space and $\phi : X \to \mathcal{C}'_c(Y)$ is a lower semi-continuous mapping such that $|\phi(x)| > 1$ for every $x \in X$, there exists a continuous map $f : X \to Y$ such that $f(x) \in \mathrm{wci}(\phi(x))$ for all $x \in X$.*

(3) *For every lower semi-continuous mapping* $\phi : X \to \mathcal{C}'_c(c_0(\lambda))$, *with* $|\phi(x)| > 1$ *for every* $x \in X$, *there exists a continuous map* $f : X \to c_0(\lambda)$ *such that* $f(x) \in \mathrm{wci}(\phi(x))$ *for all* $x \in X$.

(4) *For every closed subspace* A *of* X *and for every two maps* $g, h : A \to c_0(\lambda)$ *such that* g *is upper semi-continuous,* h *is lower semi-continuous and* $g < h$, *there exists a continuous map* $f : X \to c_0(\lambda)$ *such that* $g < f|_A < h$.

From one hand, Theorem 4.3 might be read as a possible extension of the Dowker-Katětov characterization of countably paracompact normal spaces [5, 14], see also [4]. From another hand, Theorem 4.3 should be compared with Michael's characterization [19, Theorem 3.1‴] of perfectly normal spaces by selections avoiding *supporting* points of convex sets. More precisely, in the Michael's terminology [19], if Y is a Banach space and $K \in \mathcal{F}_c(Y)$, then a *supporting* set of K is a closed convex subset S of K, $S \neq K$, such that if an interior point of a segment in K is in S, then the whole segment is in S. The set of all elements of K which are not in any supporting set of K is denoted by $\mathrm{I}(K)$ (suggesting "Inside of K"). Finally, as in [19], one may consider

$$\mathcal{D}(Y) = \left\{B \in 2^Y : B \text{ is convex and } \mathrm{I}(\mathrm{cl}_Y B) \subseteq B\right\}.$$

It is well known (see [19]) that $\mathcal{F}_c(Y) \subset \mathcal{D}(Y)$; that every convex $B \in 2^Y$ with a non-empty interior belongs to $\mathcal{D}(Y)$; and that every finite-dimensional convex $B \in 2^Y$ belongs to $\mathcal{D}(Y)$.

As for our weak convex interior, it is clear that $\mathrm{I}(K) \subseteq \mathrm{wci}(K)$ for every $K \in \mathcal{F}_c(Y)$ but the converse is not true. In fact, the Michael's [19, Theorem 3.1‴] states that a space X is perfectly normal if and only if for every separable Banach space Y, every lower semi-continuous $\phi : X \to \mathcal{D}(Y)$ has a continuous selection.

Our next result present another possible characterization of perfectly normal spaces in terms of selections.

Theorem 4.4. *Let* λ *be an infinite cardinal. For a space* X *the following conditions are equivalent*:

(1) X *is perfectly normal and* λ*-collectionwise normal.*

(2) *Whenever* Y *is a generalized* $c_0(\lambda)$*-space, every lower semi-continuous mapping* $\phi : X \to \mathcal{C}'_c(Y)$ *has a continuous selection* f *such that* $f(x) \in \mathrm{wci}(\phi(x))$ *for every* $x \in X$ *with* $|\phi(x)| > 1$.

(3) *Every lower semi-continuous mapping* $\phi : X \to \mathcal{C}'_c(c_0(\lambda))$ *has a continuous selection* f *such that* $f(x) \in \mathrm{wci}(\phi(x))$ *for every* $x \in X$ *with* $|\phi(x)| > 1$.

(4) *For every closed subspace* A *of* X *and for every two maps* $g, h : A \to c_0(\lambda)$ *such that* g *is upper semi-continuous,* h *is lower semi-continuous and* $g \leq h$, *there exists a continuous map* $f : X \to c_0(\lambda)$ *such that* $g \leq f|_A \leq h$ *and* $g(x) < f(x) < h(x)$ *whenever* $x \in A$ *with* $g(x) < h(x)$.

Returning back to Theorem 4.3, a word should be said about condition (2) of this theorem. In fact, the reader may wonder if this condition holds for all Banach spaces. The authors do not know if this is true, which suggests the following natural question.

Problem 4.5. Let X be a countably paracompact and λ-collectionwise normal space for some infinite cardinal λ, Y be a Banach space with $w(Y) \leq \lambda$, and $\phi : X \to \mathcal{C}'_c(Y)$ be lower semi-continuous such that $|\phi(x)| > 1$ for every $x \in X$.

Does there exist a continuous map $f : X \to Y$ such that $f(x) \in \mathrm{wci}(\phi(x))$ for every $x \in X$?

5. Characterizations of expandable spaces

Let λ be an infinite cardinal. A space X is called *λ-expandable* (resp., *almost λ-expandable*) if every locally finite family $\mathcal{F}$ of closed sets in X, with $|\mathcal{F}| \le \lambda$, has a locally finite (resp., point-finite) open expansion (cf. [16, 27]). We state the results, then proceed to the proofs.

Theorem 5.1. *For an infinite cardinal λ, the following conditions on a space X are equivalent*:

1. *X is λ-expandable.*
2. *For every completely metrizable space Y, with $w(Y) \le \lambda$, and every upper semi-continuous mapping $\phi : X \to \mathcal{C}(Y)$, there exist two mappings $\varphi, \psi : X \to \mathcal{C}(Y)$ such that φ is lower semi-continuous, ψ is upper semi-continuous and $\phi \subseteq \varphi \subseteq \psi$.*
3. *There exists a space Y and a locally finite family $\mathcal{G}$ of non-empty open sets in Y, with $|\mathcal{G}| = \lambda$, such that for every upper semi-continuous mapping $\phi : X \to \mathcal{C}(Y)$, there exist two mappings $\varphi, \psi : X \to \mathcal{C}(Y)$ such that φ is lower semi-continuous, ψ is upper semi-continuous and $\phi \subseteq \varphi \subseteq \psi$.*
4. *For every upper semi-continuous map $f : X \to c_0(\lambda)$, there exist two maps $g, h : X \to c_0(\lambda)$ such that g is lower semi-continuous, h is upper semi-continuous and $f \le g \le h$.*

Theorem 5.2. *For an infinite cardinal λ, the following conditions on a space X are equivalent*:

1. *X is almost λ-expandable.*
2. *For every completely metrizable space Y, with $w(Y) \le \lambda$, and every upper semi-continuous mapping $\phi : X \to \mathcal{C}(Y)$, there exists a lower semi-continuous mapping $\varphi : X \to \mathcal{C}(Y)$ such that $\phi \subseteq \varphi$.*
3. *There exists a space Y and a locally finite family $\mathcal{G}$ of non-empty open sets in Y, with $|\mathcal{G}| = \lambda$, such that for every upper semi-continuous mapping $\phi : X \to \mathcal{C}(Y)$, there exists a lower semi-continuous mapping $\varphi : X \to \mathcal{C}(Y)$ such that $\phi \subseteq \varphi$.*
4. *For every upper semi-continuous map $f : X \to c_0(\lambda)$, there exists a lower semi-continuous map $g : X \to c_0(\lambda)$ such that $f \le g$.*

Miyazaki [21] has proven the equivalence (1) and (2) in Theorem 5.1 assuming that X is normal, and has shown that every metacompact space satisfies (2) in Theorem 5.2.

It is known ([16]) that a space X is ω-expandable if and only if it is countably paracompact. Hence, by the definitions, a space X is λ-collectionwise normal and countably paracompact if and only if X satisfies one of the following two conditions: (i) X is λ-expandable and $X \in \lambda\text{-}\mathcal{PN}$; (ii) X is almost λ-expandable and $X \in \lambda\text{-}\mathcal{PN}$. Thus, we get several characterizations of a λ-collectionwise normal and countably paracompact space by combining one of the conditions in Theorems 5.1 and 5.2 with one of the conditions (1)–(9) in Theorem 3.1 and Remark 3.2. In particular, we have the following consequence which is a mapping analogue of the

Dowker's characterization [6] of collectionwise normal and countably paracompact spaces.

Corollary 5.3. *For an infinite cardinal λ, the following conditions on a normal space X are equivalent*:

(1) *X is λ-collectionwise normal and countably paracompact.*
(2) *For every upper semi-continuous map $g : X \to c_0(\lambda)$, there exists a continuous map $f : X \to c_0(\lambda)$ such that $g \leq f$.*

For other characterizations of collectionwise normal countably paracompact spaces, see [21].

We complete this paper with the following characterization of paracompact spaces which is just like Corollary 5.3, only it deals with maps to $C_0(\lambda)$, where λ is the space of all ordinals less than λ with the usual order topology.

Theorem 5.4. *For an infinite cardinal λ, the following conditions on a normal space X are equivalent*:

(1) *X is λ-paracompact.*
(2) *For every space Y, with $w(Y) \leq \lambda$, and for every upper semi-continuous map $g : X \to C_0(Y)$, there exists a continuous map $f : X \to C_0(Y)$ such that $g \leq f$.*
(3) *For every upper semi-continuous map $g : X \to C_0(\lambda)$, there exists a continuous map $f : X \to C_0(\lambda)$ such that $g \leq f$.*

In the proof of Theorem 5.4, the normality of X is only used to apply Michael's result in the implication (1) $\Rightarrow$ (2). Thus, we have the following corollary.

Corollary 5.5. *The following conditions on a Hausdorff space X are equivalent*:

(1) *X is paracompact.*
(2) *For every space Y and every upper semi-continuous map $g : X \to C_0(Y)$, there exists a continuous map $f : X \to C_0(Y)$ such that $g \leq f$.*
(3) *For every infinite cardinal λ and every upper semi-continuous map $g : X \to C_0(\lambda)$, there exists a continuous map $f : X \to C_0(\lambda)$ such that $g \leq f$.*

Concerning the statements of Corolary 5.5, the following question naturally arises.

Problem 5.6. Is a space X paracompact provided for every space Y and every two maps $g, h : X \to C_0(Y)$ such that g is upper semi-continuous, h is lower semi-continuous and $g \leq h$, there exists a continuous map $f : X \to C_0(Y)$ with $g \leq f \leq h$?

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(V. Gutev) School of Mathematical and Statistical Sciences, Faculty of Science, University of Natal, King George V Avenue, Durban 4041, South Africa
E-mail address: `gutev@nu.ac.za`

(H. Ohta) Faculty of Education, Shizuoka University, Ohya, Shizuoka, 422-8529 Japan
E-mail address: `echohta@ipc.shizuoka.ac.jp`

(K. Yamazaki) Institute of Mathematics, University of Tsukuba, Tsukuba-shi, Ibaraki, 305-8571 Japan
E-mail address: `kaori@math.tsukuba.ac.jp`

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THE BAIRE SPACE ORDERED BY EVENTUAL DOMINATION: SPECTRA

JÖRG BRENDLE*

Abstract. These are notes of the author's talk on various types of spectra associated naturally with the eventually domination ordering on the Baire space ω^ω, given at the *General Topology Symposium* at Kobe University in December 2002. The report comes in two parts: in the first half, we present an outline of the lecture, giving ideas of some of the arguments without going too deeply into details. The second part presents the technical niceties of some proofs. This part was circulated previously under the title *Chubu Marginalia* [2].

1. Outline of the lecture

The *Baire space* ω^ω is the set of all functions from the natural numbers ω to ω, equipped with the product topology of the discrete topology. Given $f, g \in \omega^\omega$ say that g *eventually dominates* f ($f \leq^* g$ in symbols) if $f(n) \leq g(n)$ holds for all but finitely many $n \in \omega$.

A family $\mathcal{F} \subseteq \omega^\omega$ is called *unbounded* if there is no $g \in \omega^\omega$ with $f \leq^* g$ for all $f \in \mathcal{F}$. $\mathcal{F} \subseteq \omega^\omega$ is said to be *dominating* if for all $g \in \omega^\omega$ there is $f \in \mathcal{F}$ with $g \leq^* f$. It is easy to see that a dominating family is also unbounded. We let $\mathfrak{b} := \min\{|\mathcal{F}|;\ \mathcal{F} \subseteq \omega^\omega \text{ unbounded}\}$, the *(un)bounding number*. $\mathfrak{d} := \min\{|\mathcal{F}|;\ \mathcal{F} \subseteq \omega^\omega \text{ dominating}\}$ is the *dominating number*. The *cardinal invariants* $\mathfrak{b}$ and $\mathfrak{d}$ characterize the combinatorial structure of $(\omega^\omega, \leq^*)$.

Fact 1.1. $\aleph_1 \leq \mathfrak{b} \leq cf(\mathfrak{d}) \leq \mathfrak{d} \leq \mathfrak{c}$ *and* $\mathfrak{b}$ *is regular.*

(Here, cf means *cofinality*, and $\mathfrak{c} = |2^\omega| = |\mathbb{R}|$ stands for the size of the continuum.)

As a leitmotiv for this talk we address: *What other notions can be used to describe the combinatorial structure of* $(\omega^\omega, \leq^*)$*?*

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For a given preorder $(P, \leq)$ (that is, $\leq$ is reflexive and transitive, but not necessarily antisymmetric), Fuchino and Soukup [4] defined the following four spectra.

(i) the *unbounded chain spectrum* $\mathfrak{S}^\uparrow(P)$, the set of all regular cardinals κ such that there is an unbounded increasing chain of length κ in P;

(ii) the *hereditarily unbounded set spectrum* $\mathfrak{S}^h(P)$, the set of all cardinals κ such that there is $A \subseteq P$ of size κ such that all subsets of A of size κ are unbounded in P while all subsets of A of size less than κ are bounded in P;

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(iii) the *unbounded set spectrum* $\mathfrak{S}(P)$, the set of all cardinals κ such that there is unbounded $A \subseteq P$ of size κ such that all subsets of A of size less than κ are bounded in P;
(iv) the *unbounded family spectrum* $\mathfrak{S}^s(P)$, the set of all cardinals κ such that there is a family $\mathcal{F} \subseteq \mathcal{P}(P)$ of size κ with $\bigcup \mathcal{F}$ being unbounded while for all $\mathcal{G} \subseteq \mathcal{F}$ of size less than κ, $\bigcup \mathcal{G}$ is bounded.

Clearly $\mathfrak{S}^{\uparrow}(P) \subseteq \mathfrak{S}^h(P) \subseteq \mathfrak{S}(P) \subseteq \mathfrak{S}^s(P)$.

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We shall study the connection between these spectra for $(P, \leq) = (\omega^\omega, \leq^*)$. It is easy to see that $\mathfrak{b} = \min \mathfrak{S}^{\uparrow}(\omega^\omega, \leq^*) = \min \mathfrak{S}^s(\omega^\omega, \leq^*)$. Fuchino and Soukup [4] asked:

Question 1.2. (Fuchino, Soukup) *Is* $\mathfrak{S}^{\uparrow}(\omega^\omega, \leq^*) \subsetneq \mathfrak{S}^h(\omega^\omega, \leq^*)$ *consistent? Is* $\mathfrak{S}^h(\omega^\omega, \leq^*) \subsetneq \mathfrak{S}(\omega^\omega, \leq^*)$ *consistent? Is* $\mathfrak{S}(\omega^\omega, \leq^*) \subsetneq \mathfrak{S}^s(\omega^\omega, \leq^*)$ *consistent?*

More concretely:

Question 1.3. *Is* $\aleph_2 \in \mathfrak{S}^h(\omega^\omega, \leq^*) \setminus \mathfrak{S}^{\uparrow}(\omega^\omega, \leq^*)$ *consistent? Is* $\aleph_2 \in \mathfrak{S}(\omega^\omega, \leq^*) \setminus \mathfrak{S}^h(\omega^\omega, \leq^*)$ *consistent? Is* $\aleph_2 \in \mathfrak{S}^s(\omega^\omega, \leq^*) \setminus \mathfrak{S}(\omega^\omega, \leq^*)$ *consistent?*

Define $\mathfrak{b}'$ to be the supremum of cofinalities of unbounded well–ordered chains in ω^ω. $\mathfrak{b}^*$ is the minimal κ such that every unbounded family $\mathcal{F} \subseteq \omega^\omega$ has an unbounded subfamily of size κ.

Fact 1.4. $\mathfrak{b}' = \sup(\mathfrak{S}^{\uparrow}(\omega^\omega, \leq^*))$, $\mathfrak{b}^* = \sup(\mathfrak{S}(\omega^\omega, \leq^*))$, *as well as* $\mathfrak{b} \leq \mathfrak{b}' \leq \mathfrak{b}^* \leq \mathfrak{d}$.

The first instance of the above question has been answered a couple of years ago in joint work with LaBerge [1].

Theorem 1.5. *It is consistent that* $\aleph_2 \in \mathfrak{S}^h(\omega^\omega, \leq^*) \setminus \mathfrak{S}^{\uparrow}(\omega^\omega, \leq^*)$ *and* $\mathfrak{c} = \aleph_2$.

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We proceed to present the main ideas of the proof of Theorem 1.5. Let A be a set. $\mathbb{C}_A$ denotes *Cohen forcing* with index set A, that is, the collection of finite partial functions $s : A \times \omega \to 2$ ordered by reverse inclusion. The latter means that $t \leq s$ iff $t \supseteq s$. For each $a \in A$, $\mathbb{C}_A$ adds a Cohen–generic real c_a. Let $X \subseteq A$. Then $\mathbb{C}_X <\circ \mathbb{C}_A$ (we say $\mathbb{C}_X$ *completely embeds into* $\mathbb{C}_A$), i.e., $\mathbb{C}_X$ is a subforcing of $\mathbb{C}_A$, and the "intermediate" generic extension via $\mathbb{C}_X$ is a submodel of the generic extension via $\mathbb{C}_A$.

As usual, $\mathbb{D}$ denotes *Hechler forcing*, that is, the collection of all pairs (s, f) where $s \in \omega^{<\omega}$, $f \in \omega^\omega$ and $s \subseteq f$. We order $\mathbb{D}$ by stipulating $(t, g) \leq (s, f)$ iff $t \supseteq s$ and $g \geq f$ everywhere. $\mathbb{D}$ generically adds a real d which eventually dominates all ground model reals.

In the extension V_X via $\mathbb{C}_X$, let $\mathbb{D}_X$ denote Hechler forcing in the sense of V_X. This means of course that $\mathbb{D}_X = \{(s, f);\ f \in V_X \cap \omega^\omega\}$.

We are ready to describe the forcing we are going to use. Let $V \models CH$. Set

$$\mathbb{P} = \mathbb{C}_{\omega_2} \star \prod_{\alpha<\omega_2}^{<\omega} \dot{\mathbb{D}}_\alpha$$

(Here, $\star$ denotes *iteration* as usual, and the superscript $^{<\omega}$ means we are forcing with the *finite support product.*) Let W be the generic extension of V via $\mathbb{P}$.

Let us list a few properties of the forcing $\mathbb{P}$ and of the resulting model W.

(a) $\mathbb{P}$ is ccc; so it preserves cardinals.
(b) As above, let c_α, $\alpha < \omega_2$, denote the Cohen reals adjoined by $\mathbb{C}_{\omega_2}$. Also let d_α, $\alpha < \omega_2$, denote the Hechler reals adjoined by $\mathbb{D}_\alpha$ (over the generic extension via $\mathbb{C}_{\omega_2}$). Then $c_\alpha \leq^* d_\beta \iff \alpha < \beta$.
(c) Let $\dot{g}$ be a $\mathbb{P}$–name for a real. Then $\Vdash_{\mathbb{P}} |\{\alpha < \omega_2;\ \dot{c}_\alpha \leq \dot{g}\}| \leq \aleph_1$ (this is so because of the way $\mathbb{P}$ factorizes as a product).
(d) In W, let $\mathcal{F} = \{c_\alpha;\ \alpha < \omega_2\}$. Then $\mathcal{F}$ witnesses $\aleph_2 \in \mathfrak{S}^h(\omega^\omega, \leq^*)$, by (b) and (c).
(e) $\aleph_2 \notin \mathfrak{S}^\uparrow(\omega^\omega, \leq^*)$. In fact, there are no well–ordered ω_2–sequences in $(\omega^\omega, \leq^*)$ in W.

We briefly describe the proof of (e) for this is the heart of the matter. It is based on the following two lemmata.

Lemma 1.6. (Kunen) *Assume CH. There are no well–ordered ω_2–sequences in $(\omega^\omega, \leq^*)$ in V_A (the extension via $\mathbb{C}_A$).*

This is proved via a standard "isomorphism–of–names" trick.

Lemma 1.7. (Interpolation trick) *Let $\mathbb{Q}$ and $\mathbb{R}$ be p.o.'s. Assume $\dot{h}_1$ and $\dot{h}_2$ are $\mathbb{Q}$–names for reals, and $\dot{g}$ is an $\mathbb{R}$–name for a real such that $(q,r) \Vdash_{\mathbb{Q}\times\mathbb{R}} \dot{h}_1 \leq^* \dot{g} \leq^* \dot{h}_2$. Then $q \Vdash_{\mathbb{Q}}$ "$\exists f \in \omega^\omega \cap V : \dot{h}_1 \leq^* f \leq^* \dot{h}_2$."*

Given a sequence $\{\dot{f}_\alpha;\ \alpha < \omega_2\}$ of $\mathbb{P}$–names use a Δ–system argument to make supports nice (this uses CH) and step into an intermediate extension (essentially a Cohen extension) such that the $\dot{f}_\alpha$ are adjoined by a product over this intermediate extension. Assume $\Vdash_{\mathbb{P}}$ "the $\dot{f}_\alpha$ are well–ordered by $\leq^*$." Using Lemma 1.7, we can interpolate reals g_α from the intermediate extension. This contradicts Lemma 1.6. □

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Recently we proved:

Theorem 1.8. *It is consistent that $\aleph_2 \in \mathfrak{S}(\omega^\omega, \leq^*) \setminus \mathfrak{S}^h(\omega^\omega, \leq^*)$ and $\mathfrak{c} = \aleph_3$.*

With more work one can do better.

Theorem 1.9. *It is consistent that $\aleph_2 \in \mathfrak{S}(\omega^\omega, \leq^*) \setminus \mathfrak{S}^h(\omega^\omega, \leq^*)$ and $\mathfrak{c} = \aleph_2$.*

The proof of Theorem 1.8 was briefly sketched at the end of our lecture. Since we include a detailed account of these results below in Section 2, we shall not give this outline here.

We conjecture that similar techniques can be used to get a model where $\aleph_2 \in \mathfrak{S}^s(\omega^\omega, \leq^*) \setminus \mathfrak{S}(\omega^\omega, \leq^*)$, but so far we have been unable to prove this.

More recently, we obtained:

Theorem 1.10. (see [3]) *It is consistent that $\mathfrak{c} = \aleph_2$, $\aleph_2 \in \mathfrak{S}(\omega^\omega, \leq^*)$ and there is no definable relation $A \subseteq (\omega^\omega)^2$ such that A well–orders some subset $\{f_\alpha \in \omega^\omega;\ \alpha < \omega_2\}$ of the reals (i.e., $\alpha < \beta$ iff $(f_\alpha, f_\beta) \in A$).*

In this model, we obviously have $\aleph_2 \notin \mathfrak{S}^\uparrow(\omega^\omega, \leq^*)$. As a matter of fact, it is not difficult to show that $\aleph_2 \in \mathfrak{S}^h(\omega^\omega, \leq^*)$ is sufficient to construct a definable well–ordering of a set of reals of length ω_2 [3]. Therefore, Theorem 1.10 is a strengthening of Theorem 1.9. See [3] for details.

2. Technical details

Assume A and B are disjoint sets and $\mathcal{G} \subseteq A \times B$ is a bipartite graph. We will define the poset $\mathbb{P} = \mathbb{P}^{\mathcal{G}}$.

For $X \subseteq A$, $\mathbb{C}_X$ denotes the algebra for adding $|X|$ many Cohen reals with index set X. Clearly, $\mathbb{C}_X <\circ \mathbb{C}_Y$ for $X \subseteq Y$. Fix $b \in B$ and let $X_b = \{a \in A;\ (a,b) \in \mathcal{G}\}$. In the generic extension via $\mathbb{C}_{X_b}$ we have Hechler forcing $\mathbb{D}$ which we denote by $\mathbb{D}_{X_b}$. The p.o. $\mathbb{P}$ we force with is a two–step iteration given by

$$\mathbb{P} = \mathbb{C}_A \star \prod_{b \in B}^{<\omega} \dot{\mathbb{D}}_{X_b}$$

Here $\prod^{<\omega}$ means that we take the finite support product of the partial orders $\mathbb{D}_{X_b}$ in the generic extension via $\mathbb{C}_A$. Conditions $p \in \mathbb{P}$ can be canonically represented by

$$p = (s_a^p, (t_b^p, \dot{h}_b^p);\ a \in F^p, b \in G^p)$$

where $F^p \subseteq A$ and $G^p \subseteq B$ are finite. So this means $s_a^p \in \mathbb{C}_{\{a\}}$ and $\Vdash_{\mathbb{C}_{X_b}} (t_b^p, \dot{h}_b^p) \in \dot{\mathbb{D}}_{X_b}$. We let $\mathsf{supp}(p) = F^p \cup G^p$, the support of p.

For later use we notice two facts.

Fact 2.1. *Given $X \subseteq A$ and $Y \subseteq B$, we form $\mathbb{P}^{\mathcal{G}}_{X,Y} = \mathbb{P}_{X,Y} = \mathbb{C}_X \star \prod_{b \in Y}^{<\omega} \dot{\mathbb{D}}_{X \cap X_b}$. Then $\mathbb{P}_{X,Y} <\circ \mathbb{P}$.*

Proof. Let $p = (s_a^p, (t_b^p, \dot{h}_b^p);\ a \in F^p, b \in G^p) \in \mathbb{P}$. We need to find a reduction $p_0 \in \mathbb{P}_{X,Y}$ of p. We shall have $p_0 = (s_a^{p_0}, (t_b^{p_0}, \dot{h}_b^{p_0});\ a \in F^{p_0}, b \in G^{p_0})$ where

- $F^{p_0} = F^p \cap X$
- $G^{p_0} = G^p \cap Y$
- $s_a^{p_0} = s_a^p$ for $a \in F^{p_0}$
- $t_b^{p_0} = t_b^p$ for $b \in G^{p_0}$

and the $\dot{h}_b^{p_0}$ are given as follows.

Write $s^p = (s_a^p;\ a \in F^p) \in \mathbb{C}_A$. So $s^{p_0} = (s_a^p;\ a \in F^{p_0}) = s^p \restriction \mathbb{C}_X \in \mathbb{C}_X$. For $b \in G^{p_0}$ let $\mathsf{supp}(\dot{h}_b^p) \subseteq X_b$ be what is needed to decide $\dot{h}_b^p$. By ccc–ness of $\mathbb{C}_{X_b}$, $\mathsf{supp}(\dot{h}_b^p)$ is countable. Set $X^0 = X \cap \bigcup_{b \in G^{p_0}} \mathsf{supp}(\dot{h}_b^p)$. Also let $X^1 = A \setminus X$. So X^0 and X^1 are disjoint, and X^0 is countable. Let $\{s_n;\ n \in \omega\}$ enumerate the conditions of $\mathbb{C}_{X^0}$ which are below $s^{p_0} \restriction \mathbb{C}_{X^0} (= s^p \restriction \mathbb{C}_{X^0})$. Recursively find numbers $\ell_{n,k,b}$ ($k \leq n$ and $b \in G^{p_0}$), and conditions $t_n \in \mathbb{C}_{X^0}$, $u_n \in \mathbb{C}_{X^1}$ such that $t_n \leq s_n$ and the u_n form a decreasing chain below $s^p \restriction \mathbb{C}_{X^1}$ and such that

$$t_n \cup u_n \Vdash \dot{h}_b^p(k) = \ell_{n,k,b} \text{ for } k \leq n \text{ and } b \in G^{p_0}.$$

This is clearly possible. Note that for $b \in G^{p_0}$, we actually have $(t_n \cup u_n) \restriction \mathsf{supp}(\dot{h}_b^p) \Vdash \dot{h}_b^p(k) = \ell_{n,k,b}$ for $k \leq n$.

Now define the $\mathbb{C}_{X \cap X_b}$–name $\dot{h}_b^{p_0}$ by stipulating that $t_n \restriction \mathsf{supp}(\dot{h}_b^p)$ forces $\dot{h}_b^{p_0}(k) = \ell_{n,k,b}$ for $k \leq n$. We need to verify this is well–defined, that is, no two compatible conditions force contradictory statements. To see this, fix k and assume $t_n \restriction \mathsf{supp}(\dot{h}_b^p)$ and $t_m \restriction \mathsf{supp}(\dot{h}_b^p)$ are compatible for some $m, n \geq k$. Since the u_j are decreasing, $(t_n \cup u_n) \restriction \mathsf{supp}(\dot{h}_b^p)$ and $(t_m \cup u_m) \restriction \mathsf{supp}(\dot{h}_b^p)$ must be compatible as well. So they must force the same value to $\dot{h}_b^p(k)$, i.e., $\ell_{n,k,b} = \ell_{m,k,b}$ as required.

This completes the definition of p_0. We leave it to the reader to verify that given $q_0 \leq p_0$, $q_0 \in \mathbb{P}_{X,Y}$, there is a common extension $q \in \mathbb{P}$ of q_0 and p. $\square$

This entails for example that if $\dot f$ and $\dot g$ are $\mathbb{P}_{X,Y}$–names, then

$$\Vdash_{\mathbb{P}} \dot f \leq^* \dot g \iff \Vdash_{\mathbb{P}_{X,Y}} \dot f \leq^* \dot g$$

Fact 2.2. *Given $X_0, X_1 \subseteq A$, $Y_0, Y_1 \subseteq B$, $X = X_0 \cap X_1$, $Y = Y_0 \cap Y_1$, and bijections $v : X_0 \to X_1$, $w : Y_0 \to Y_1$ fixing X and Y respectively,* if $(a,b) \in \mathcal{G}$ *iff $(v(a), w(b)) \in \mathcal{G}$ for all $a \in X_0$ and all $b \in Y_0$,* then $\mathbb{P}_{X_0,Y_0}$ *and* $\mathbb{P}_{X_1,Y_1}$ *are canonically isomorphic via an isomorphism $\phi = \phi^{X_0,Y_0}_{X_1,Y_1}$ fixing $\mathbb{P}_{X,Y}$.*

Note this means that if $\dot f$ is a $\mathbb{P}_{X,Y}$–name for a real, $\dot g$ is a $\mathbb{P}_{X_0,Y_0}$–name for a real and $p \in \mathbb{P}_{X_0,Y_0}$, $k \in \omega$ are such that

$$p \Vdash_{X_0,Y_0} \forall n \geq k\ (\dot f(n) \leq \dot g(n))$$

then if $\phi(\dot g)$ is the image of the name $\dot g$ we have

$$\phi(p) \Vdash_{X_1,Y_1} \forall n \geq k\ (\dot f(n) \leq \phi(\dot g)(n))$$

Here, $\phi(p)$ is gotten from p as follows. $F^{\phi(p)} = v(F^p)$, $G^{\phi(p)} = w(G^p)$, $s_a^{\phi(p)} = s^p_{v^{-1}(a)}$ for $a \in F^{\phi(p)}$, $t_b^{\phi(p)} = t^p_{w^{-1}(b)}$ for $b \in G^{\phi(p)}$, and $\dot h_b^{\phi(p)}$ is the image of $\dot h^p_{w^{-1}(b)}$ under the isomorphism between $\mathbb{C}_{X_0}$ and $\mathbb{C}_{X_1}$ induced by v, for all $b \in G^{\phi(p)}$. Furthermore p and $\phi(p)$ are compatible (in $\mathbb{P}_{X_0 \cup X_1, Y_0 \cup Y_1}$). (*may not need this??? details see handwritten notes*)

Denote the Cohen reals added by $\mathbb{C}_A$ by $\{c_a;\ a \in A\}$, and the Hechler reals adjoined by the finite support product, by $\{d_b;\ b \in B\}$.

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Proof of Theorem 1.8. Assume *GCH*. Let A and B be sets of size $\aleph_3$, without loss $A = B = \omega_3$. Recursively construct a bipartite graph $\mathcal{G} \subseteq A \times B$ such that

(i) for all countable $Y \subseteq B$ there is $a \in \omega_2$ such that $(a,b) \notin \mathcal{G}$ for all $b \in Y$

(ii) for all $X \subseteq A$ with $X \subseteq \omega_2$ and $|X| \leq \aleph_1$, there is $b \in B$ such that $(a,b) \in \mathcal{G}$ for all $a \in X$

(iii) for all pairwise disjoint countable $X, X_\alpha \subseteq A$ $(\alpha < \omega_2)$, all pairwise disjoint countable $Y, Y_\alpha \subseteq B$ $(\alpha < \omega_2)$, all $\{x_n;\ n \in \omega\} \subseteq 2^{X \cup \bigcup_{\alpha<\omega_2} X_\alpha}$, all $\{y_n;\ n \in \omega\} \subseteq 2^{Y \cup \bigcup_{\alpha<\omega_2} Y_\alpha}$, and all bipartite graphs $\mathcal{G}_0 \subseteq \omega \times \omega$, there are $\Omega \subseteq \omega_2$ of size $\aleph_2$ as well as $\{a_n;\ n \in \omega\} \subseteq A$ disjoint from X, X_α and $\{b_n;\ n \in \omega\} \subseteq B$ disjoint from Y, Y_α such that

$$(a, b_n) \in \mathcal{G} \iff x_n(a) = 1$$

for all $a \in X \cup \bigcup_{\alpha\in\Omega} X_\alpha$ and all $n \in \omega$,

$$(a_n, b) \in \mathcal{G} \iff y_n(b) = 1$$

for all $b \in Y \cup \bigcup_{\alpha\in\Omega} Y_\alpha$ and all $n \in \omega$ and

$$(a_n, b_m) \in \mathcal{G} \iff (n,m) \in \mathcal{G}_0$$

for all $n, m \in \omega$.

This is done by a recursive construction of length ω_3, taking care of (ii) in the first step and of (iii) in the remaining steps. Since we are allowed to thin out to a set Ω we can guarantee that (i) stays valid along the construction. For example we could stipulate that for all $b \in B$, the set $X_b = \{a \in \omega_2;\ (a,b) \in \mathcal{G}\}$ is non–stationary in ω_2. Details are left to the reader.

It is clear by $|A| = |B| = \aleph_3$ that the size of the continuum will be $\aleph_3$.

Lemma 2.3. *The Cohen reals* $\{c_a;\ a < \omega_2\}$ *are unbounded in the* $\mathbb{P}$*–extension, yet every subfamily of size* $\aleph_1$ *is bounded.*

Proof. Let $\dot g$ be a $\mathbb{P}$–name for a real. There are $X \subseteq A$ countable and $Y \subseteq B$ countable such that $\dot g$ is a $\mathbb{P}_{X,Y}$–name. By (i) there is $a \in \omega_2 \setminus \bigcup_{b \in Y} X_b$. Note that for such a, $\mathbb{P}_{X \cup \{a\}, Y} = \mathbb{P}_{X,Y} \times \mathbb{C}_{\{a\}}$. That is, c_a is still Cohen over the extension via $\mathbb{P}_{X,Y}$ and, in particular, it is not bounded by the interpretation of $\dot g$.

That every subfamily of size $\aleph_1$ is bounded immediately follows from property (ii). □

Main Lemma 2.4. *In the* $\mathbb{P}$*–extension, the following holds: Assume* $\{f_\alpha;\ \alpha < \omega_2\}$ *is such that for all* $\beta < \omega_2$, $\{f_\alpha;\ \alpha < \beta\}$ *is bounded. Then there is* $\Omega \subseteq \omega_2$, $|\Omega| = \aleph_2$, *such that* $\{f_\alpha;\ \alpha \in \Omega\}$ *is bounded.*

Proof. Let $\{\dot f_\alpha;\ \alpha < \omega_2\}$ and $\{\dot g_\beta;\ \beta < \omega_2\}$ be $\mathbb{P}$–names such that

$$\Vdash_{\mathbb{P}} \dot f_\alpha \leq^* \dot g_\beta$$

for all $\alpha < \beta < \omega_2$. By CH and a Δ–system argument we may assume that there are pairwise disjoint countable sets $X, X_\alpha, A_\beta \subseteq A$ ($\alpha, \beta < \omega_2$) and pairwise disjoint countable sets $Y, Y_\alpha, B_\beta \subseteq B$ ($\alpha, \beta < \omega_2$) such that all $\dot f_\alpha$ are $\mathbb{P}_{X \cup X_\alpha, Y \cup Y_\alpha}$–names and all $\dot g_\beta$ are $\mathbb{P}_{X \cup A_\beta, Y \cup B_\beta}$–names. By the remark after Fact 2.1,

$$\Vdash_{\mathbb{P}_{X \cup X_\alpha \cup A_\beta, Y \cup Y_\alpha \cup B_\beta}} \dot f_\alpha \leq^* \dot g_\beta$$

for all $\alpha < \beta < \omega_2$.

We may further assume that for $\alpha \neq \beta$ we have bijections $v_{\alpha,\beta} : X \cup A_\alpha \to X \cup A_\beta$ and $w_{\alpha,\beta} : Y \cup B_\alpha \to Y \cup B_\beta$ fixing X and Y respectively such that for all $a \in X \cup A_\alpha$ and all $b \in Y \cup B_\alpha$, we have $(a,b) \in \mathcal{G}$ iff $(v_{\alpha,\beta}(a), w_{\alpha,\beta}(b)) \in \mathcal{G}$. By Fact 2.2, this means that we get an isomorphism $\phi = \phi^\alpha_\beta = \phi^{X \cup A_\alpha, Y \cup B_\alpha}_{X \cup A_\beta, Y \cup B_\beta}$ between $\mathbb{P}_{X \cup A_\alpha, Y \cup B_\alpha}$ and $\mathbb{P}_{X \cup A_\beta, Y \cup B_\beta}$. We may also suppose that this induced isomorphism identifies the corresponding names $\dot g_\alpha$ and $\dot g_\beta$, i.e., $\phi(\dot g_\alpha) = \dot g_\beta$. List $A_\alpha = \{a_{\alpha,n};\ n \in \omega\}$ such that $v_{\alpha,\beta}(a_{\alpha,n}) = a_{\beta,n}$ and $B_\alpha = \{b_{\alpha,n};\ n \in \omega\}$ such that $w_{\alpha,\beta}(b_{\alpha,n}) = b_{\beta,n}$. Let $\mathcal{G}_0$ be such that $(n,m) \in \mathcal{G}_0$ iff $(a_{\alpha,n}, b_{\alpha,m}) \in \mathcal{G}$ (note this is independent of the choice of α).

Next, for each α there is a cofinal set $C_\alpha \subseteq \omega_2 \setminus (\alpha+1)$ such that for all $\beta, \beta' \in C_\alpha$: $(a,b) \in \mathcal{G}$ iff $(a, w_{\beta,\beta'}(b)) \in \mathcal{G}$ for all $a \in X_\alpha$ and all $b \in B_\beta$, and $(a,b) \in \mathcal{G}$ iff $(v_{\beta,\beta'}(a), b) \in \mathcal{G}$ for all $a \in A_\beta$ and all $b \in Y_\alpha$. Define $x_n : X \cup \bigcup_{\alpha < \omega_2} X_\alpha \to 2$ by

$$x_n(a) = \begin{cases} 1 & \text{if } (a \in X \text{ and } (a, b_{\beta,n}) \in \mathcal{G}) \text{ or} \\ & (a \in X_\alpha \text{ and } (a, b_{\beta,n}) \in \mathcal{G} \text{ for } \beta \in C_\alpha) \\ 0 & \text{otherwise} \end{cases}$$

and $y_n : Y \cup \bigcup_{\alpha < \omega_2} Y_\alpha \to 2$ by

$$y_n(b) = \begin{cases} 1 & \text{if } (b \in Y \text{ and } (a_{\beta,n}, b) \in \mathcal{G}) \text{ or} \\ & (b \in Y_\alpha \text{ and } (a_{\beta,n}, b) \in \mathcal{G} \text{ for } \beta \in C_\alpha) \\ 0 & \text{otherwise} \end{cases}$$

(again this is independent of the choice of β).

We may now find $\Omega \subseteq \omega_2$ of size $\aleph_2$ and $A_{\omega_2} = \{a_n;\ n \in \omega\} \subseteq A$, $B_{\omega_2} = \{b_n;\ n \in \omega\} \subseteq B$ satisfying the requirements of clause (iii). Note this means that for any $\alpha \in \Omega$ and any $\beta \in C_\alpha$ we have functions $v_{\beta,\omega_2} : X \cup X_\alpha \cup A_\beta \to X \cup X_\alpha \cup A_{\omega_2}$

and $w_{\beta,\omega_2} : Y \cup Y_\alpha \cup B_\beta \to Y \cup Y_\alpha \cup B_{\omega_2}$ fixing $X \cup X_\alpha$ and $Y \cup Y_\alpha$ respectively such that

$$(a,b) \in \mathcal{G} \iff (v_{\beta,\omega_2}(a), w_{\beta,\omega_2}(b)) \in \mathcal{G}$$

for all $a \in X \cup X_\alpha \cup A_\beta$ and all $b \in Y \cup Y_\alpha \cup B_\beta$. By Fact 2.2 we get a canonical isomorphism $\phi^\beta_{\omega_2} : \mathbb{P}_{X\cup X_\alpha\cup A_\beta, Y\cup Y_\alpha\cup B_\beta} \to \mathbb{P}_{X\cup X_\alpha\cup A_{\omega_2}, Y\cup Y_\alpha\cup B_{\omega_2}}$. Let $\dot g_{\omega_2}$ be the $\phi^\beta_{\omega_2}$–image of $\dot g_\beta$ (note this is independent of the choice of β). By the discussion after Fact 2.2 and the assumption on $\dot f_\alpha$ and $\dot g_\beta$, we see that $\Vdash_{\mathbb{P}_{X\cup X_\alpha\cup A_{\omega_2}, Y\cup Y_\alpha\cup B_{\omega_2}}} \dot f_\alpha \leq^* \dot g_{\omega_2}$. By the remark after Fact 2.1, this is also true in the $\mathbb{P}$–generic extension. □

□

♣♣♣

Proof of Theorem 1.9. Assume CH. Construct first a bipartite graph $\mathcal{H} \subseteq \omega_1 \times \omega_1$ such that

(i) for all countable $Y \subseteq \omega_1$ there is $x \in \omega_1$ such that $(x,y) \notin \mathcal{H}$ for all $y \in Y$

(ii) for all countable $X = \{x_n;\ n \in \omega\} \subseteq \omega_1$, all countable $Y \subseteq \omega_1$ and all $f : \omega \times X \times \omega \to 2$ there are countable $X' = \{x'_{n,m};\ n,m \in \omega\} \subseteq \omega_1$ disjoint from X and countable $Y' = \{y'_m;\ m \in \omega\} \subseteq \omega_1$ disjoint from Y such that $(x_n, y'_m) \in \mathcal{H}$ for all $n,m \in \omega$,

$$(x'_{n,m}, y) \in \mathcal{H} \iff (x_n, y) \in \mathcal{H}$$

for all $n,m \in \omega$ and $y \in Y$, as well as

$$(x'_{n,m}, y'_\ell) \in \mathcal{H} \iff f(m, x_n, \ell) = 1$$

for all $n,m,\ell \in \omega$.

Using CH, $\mathcal{H}$ can be constructed by an easy recursion of length ω_1, taking care of (i) and (ii) alternately.

Next let $A = \omega_2 \times \omega_1$ and $B = \omega_2 \dot\cup \omega_1$. $\mathcal{G} \subseteq A \times B$ is the bipartite graph given by

$$((\zeta,\eta),\zeta') \in \mathcal{G} \iff \zeta < \zeta'$$
$$((\zeta,\eta),\eta') \in \mathcal{G} \iff (\eta,\eta') \in \mathcal{H}$$

for all $\zeta,\zeta' \in \omega_2$ and $\eta,\eta' \in \omega_1$. We consider forcing with $\mathbb{P} = \mathbb{P}^{\mathcal{G}}$ as before. Since $|A| = |B| = \aleph_2$, it is immediate that the size of the continuum will be $\aleph_2$.

By (i) and definition of $\mathcal{G}$, properties analogous to (i) and (ii) of the previous proof hold and it follows that the Cohen reals witness that $\aleph_2 \in \mathfrak{S}(\omega^\omega, \leq^*)$. So we are left with showing:

Main Lemma 2.5. *In the $\mathbb{P}$–extension, the following holds: Assume $\{f_\alpha;\ \alpha < \omega_2\}$ is such that for all $\beta < \omega_2$, $\{f_\alpha;\ \alpha < \beta\}$ is bounded. Then there is $\Omega \subseteq \omega_2$, $|\Omega| = \aleph_2$, such that $\{f_\alpha;\ \alpha \in \Omega\}$ is bounded.*

Proof. As in the previous proof we have $\dot f_\alpha$, $\dot g_\beta$ and countable pairwise disjoint sets $X, X_\alpha, A_\beta \subseteq A$ as well as countable pairwise disjoint sets $Y, Y_\alpha, B_\beta \subseteq B$, $\alpha,\beta < \omega_2$. Let $Y^0 = Y \cap \omega_2$ and let $Y^1 = Y \cap \omega_1$, the two disjoint pieces Y is made off. By enlarging supports, if necessary, we may further assume there are $\theta < \omega_1$ and $\zeta_\alpha, \xi_\alpha, \alpha < \omega_2$, such that $X = X^0 \times \theta$, $X_\alpha = X^0_\alpha \times \theta$, $A_\beta = A^0_\beta \times \theta$, $Y^1 \subseteq \theta$, $Y_\alpha, B_\beta \subseteq \omega_2$, and $\sup(X^0) < \zeta_0 < ... < \zeta_\alpha < \min(X^0_\alpha) < \sup(X^0_\alpha) < \xi_\alpha < \min(A^0_\alpha) < \sup(A^0_\alpha) < \zeta_{\alpha+1}...$ and $\sup(Y^0) < \zeta_0 < ... < \zeta_\alpha < \min(Y_\alpha) < \sup(Y_\alpha) < \xi_\alpha < \min(B_\alpha) < \sup(B_\alpha) < \zeta_{\alpha+1}...$ This means that for all $\alpha < \beta$, all

$a \in X \cup X_\alpha$ and all $b \in B_\beta$ we have $(a,b) \in \mathcal{G}$ and for all $a \in A_\beta$ and all $b \in Y^0 \cup Y_\alpha$ we have $(a,b) \notin \mathcal{G}$. List $\theta = \{x_n;\ n \in \omega\}$. Also list $A_\beta = \{(a_{\beta,m}, x_n);\ n,m \in \omega\}$ (so $A^0_\beta = \{a_{\beta,m};\ m \in \omega\}$) and $B_\beta = \{b_{\beta,m};\ m \in \omega\}$ in such that way that $((a_{\alpha,m}, x_n), b_{\alpha,\ell}) \in \mathcal{G}$ iff $((a_{\beta,m}, x_n), b_{\beta,\ell}) \in \mathcal{G}$, for all α, β and all n, m, ℓ (note that by definition of $\mathcal{G}$, this depends only on m and ℓ and not on n; this is irrelevant, however).

Apply (ii) with the X there being $\{x_n;\ n \in \omega\} = \theta$, the Y there being $Y^1 \subseteq \theta$ and f given by $f(m, x_n, \ell) = 1$ iff $((a_{\beta,m}, x_n), b_{\beta,\ell}) \in \mathcal{G}$. We get $X' = \{x'_{n,m};\ n,m \in \omega\}$ and $B_{\omega_2} = Y' = \{y'_m;\ m \in \omega\}$ as stipulated in (ii). Choose any $a_0 > \zeta_0$ from ω_2. Let $A_{\omega_2} = \{a_0\} \times X'$. For $\alpha < \beta$, define $v = v_{\beta,\omega_2} : X \cup X_\alpha \cup A_\beta \to X \cup X_\alpha \cup A_{\omega_2}$ and $w = w_{\beta,\omega_2} : Y \cup B_\beta \to Y \cup B_{\omega_2}$ by $v(a_{\beta,m}, x_n) = (a, x'_{n,m})$ and $w(b_{\beta,m}) = y'_m$ (identity otherwise). Then we have $(a,b) \in \mathcal{G}$ iff $(v(a), w(b)) \in \mathcal{G}$ for all $a \in X \cup X_\alpha \cup A_\beta$ and all $b \in Y \cup B_\beta$ and all $\alpha < \beta$. (*check details!*) By Fact 2.2, this means we get an isomorphism $\phi = \phi^\beta_{\omega_2} : \mathbb{P}_{X \cup X_\alpha \cup A_\beta, Y \cup B_\beta} \to \mathbb{P}_{X \cup X_\alpha \cup A_{\omega_2}, Y \cup B_{\omega_2}}$. As in the previous proof, let $\dot g_{\omega_2}$ be the $\phi^\beta_{\omega_2}$–image of $\dot g_\beta$. The following claim finishes the proof.

Main Claim 2.6. *Given $p \in \mathbb{P}$ and $\alpha_0 < \omega_2$, there are $q \leq p$ and $\alpha \geq \alpha_0$ with $p \Vdash \dot g_{\omega_2} \geq^* \dot f_\alpha$.*

Proof. Let $p \in \mathbb{P}$ and $\alpha_0 < \omega_2$. Fix $\alpha \geq \alpha_0$ such that $\mathsf{supp}(p) \cap Y_\alpha = \emptyset$. We need to find $q \leq p$ and $k \in \omega$ such that $q \Vdash$ "$\dot g_{\omega_2}(n) \geq \dot f_\alpha(n)$ for all $n \geq k$."

First note we may assume $p \in \mathbb{P}_{X \cup X_\alpha \cup A_{\omega_2}, Y \cup B_{\omega_2}}$. The point is that under this assumption we will find $q \in \mathbb{P}_{X \cup X_\alpha \cup A_{\omega_2}, Y \cup Y_\alpha \cup B_{\omega_2}}$. Now, if p is arbitrary, we may first consider its reduction $\bar p$ to $\mathbb{P}_{X \cup X_\alpha \cup A_{\omega_2}, Y \cup Y_\alpha \cup B_{\omega_2}}$ which actually must belong to $\mathbb{P}_{X \cup X_\alpha \cup A_{\omega_2}, Y \cup B_{\omega_2}}$. Thus we get $\bar q \leq \bar p$ as required with $\bar q \in \mathbb{P}_{X \cup X_\alpha \cup A_{\omega_2}, Y \cup Y_\alpha \cup B_{\omega_2}}$. So $\bar q$ and p have a common extension.

Let $p^* = \phi^{-1}(p) \in \mathbb{P}_{X \cup X_\alpha \cup A_\beta, Y \cup B_\beta}$ where $\beta > \alpha$ is arbitrary. There are $q^* \leq p^*$, $q^* \in \mathbb{P}_{X \cup X_\alpha \cup A_\beta, Y \cup Y_\alpha \cup B_\beta}$, and $k \in \omega$ such that $q^* \Vdash$ "$\dot g_\beta(n) \geq \dot f_\alpha(n)$ for all $n \geq k$."

The main technical difficulty with this proof now is that $\mathbb{P}_{X \cup X_\alpha \cup A_\beta, Y \cup Y_\alpha \cup B_\beta}$ need not be isomorphic to $\mathbb{P}_{X \cup X_\alpha \cup A_{\omega_2}, Y \cup Y_\alpha \cup B_{\omega_2}}$, so we cannot go back directly. Instead, we shall *interpolate* a name $\dot h_\alpha$ between $\dot f_\alpha$ and $\dot g_\beta$ in such a way that $\dot h_\alpha$ does not depend on Y_α nor on β.

The crux of the matter is that $\mathbb{P}_{X \cup X_\alpha \cup A_\beta, Y \cup Y_\alpha \cup B_\beta}$ is the *amalgamation* of $\mathbb{P}_{X \cup X_\alpha, Y \cup Y_\alpha}$ and $\mathbb{P}_{X \cup X_\alpha \cup A_\beta, Y \cup B_\beta}$ over $\mathbb{P}_{X \cup X_\alpha, Y}$. This is so because $(a,b) \notin \mathcal{G}$ for all $a \in A_\beta$ and all $b \in Y_\alpha$ which means that the Hechler reals adjoined by Y_α do not depend on the Cohen reals adjoined by A_β.

To be more explicit, let $\bar q^*$ be the projection of q^* to $\mathbb{P}_{X \cup X_\alpha \cup A_\beta, Y \cup B_\beta}$. Next, let $\bar q^*_0$ be the projection of $\bar q^*$ to $r.o.(\mathbb{P}_{X \cup X_\alpha, Y})$ (note that, since we are working with projections in cBa's here, $\bar q^*$ does not necessarily belong to $\mathbb{P}_{X \cup X_\alpha, Y}$, but this is irrelevant here). Finally, define $q^*_0 \in r.o.(\mathbb{P}_{X \cup X_\alpha, Y \cup Y_\alpha})$ as follows. $q^*_0 \restriction r.o.(\mathbb{P}_{X \cup X_\alpha, Y}) = \bar q^*_0$ and $q^*_0 \restriction Y_\alpha = q^* \restriction Y_\alpha$. The latter means that $G^{q^*_0} \cap Y_\alpha = G^{q^*} \cap Y_\alpha$ and $(t^{q^*_0}_b, \dot h^{q^*_0}_b) = (t^{q^*}_b, \dot h^{q^*}_b)$ for $b \in G^{q^*_0} \cap Y_\alpha$. This makes sense because all such $\dot h^{q^*}_b$ are $\mathbb{C}_{(X \cup X_\alpha \cup A_\beta) \cap X_b}$–names and therefore, as $A_\beta \cap X_b = \emptyset$, $\mathbb{C}_{(X \cup X_\alpha) \cap X_b}$–names. Finally notice that q^*_0 is the projection of q^* to $r.o.(\mathbb{P}_{X \cup X_\alpha, Y \cup Y_\alpha})$.

Step for a moment into the generic extension W via $\mathbb{P}_{X \cup X_\alpha, Y}$ such that $\bar q^*_0$ belongs to the generic filter. In W, forcing with $\mathbb{P}_{X \cup X_\alpha \cup A_\beta, Y \cup Y_\alpha \cup B_\beta}$ is nothing but forcing with the product of $\mathbb{P}_{X \cup X_\alpha, Y \cup Y_\alpha}$ and $\mathbb{P}_{X \cup X_\alpha \cup A_\beta, Y \cup B_\beta}$. (So $\mathbb{P}_{X \cup X_\alpha \cup A_\beta, Y \cup Y_\alpha \cup B_\beta}$ is indeed the amalgamation as claimed above.) This means we may think of q^* as the pair $(q_0^*, \bar{q}^*)$. Since

$$(q_0^*, \bar{q}^*) \Vdash \text{“}\dot{g}_\beta(n) \geq \dot{f}_\alpha(n) \text{ for all } n \geq k,\text{”}$$

since $\dot{f}_\alpha$ is a $\mathbb{P}_{X\cup X_\alpha, Y\cup Y_\alpha}$–name, and since $\dot{g}_\beta$ is a $\mathbb{P}_{X\cup X_\alpha\cup A_\beta, Y\cup B_\beta}$–name, we may find $h_\alpha \in W \cap \omega^\omega$ such that

$$q_0^* \Vdash \text{“}h_\alpha(n) \geq \dot{f}_\alpha(n) \text{ for all } n \geq k\text{”}$$

and

$$\bar{q}^* \Vdash \text{“}\dot{g}_\beta(n) \geq h_\alpha(n) \text{ for all } n \geq k.\text{”}$$

(Simply let $h_\alpha(n) = \sup\{m;$ there is $r_0^* \leq q_0^*$ in $\mathbb{P}_{X\cup X_\alpha, Y\cup Y_\alpha}$ such that $r_0^* \Vdash \dot{f}_\alpha(n) = m\}$.)

Back in the ground model, let $\dot{h}_\alpha$ be a $\mathbb{P}_{X\cup X_\alpha, Y}$–name for h_α. Let $\bar{q} = \phi(\bar{q}^*)$. Since ϕ fixes $\mathbb{P}_{X\cup X_\alpha, Y}$, $\bar{q} \leq \bar{q}_0^*$ and $\phi(\dot{h}_\alpha) = \dot{h}_\alpha$. So we get (see the discussion after Fact 2.2)

$$\bar{q} \Vdash \text{“}\dot{g}_{\omega_2}(n) \geq \dot{h}_\alpha(n) \text{ for all } n \geq k.\text{”}$$

In $\mathbb{P}_{X\cup X_\alpha\cup A_{\omega_2}, Y\cup Y_\alpha\cup B_{\omega_2}}$, define q such that $q{\restriction}\mathbb{P}_{X\cup X_\alpha\cup A_{\omega_2}, Y\cup B_{\omega_2}} = \bar{q}$ and $q{\restriction}Y_\alpha = q_0^*{\restriction}Y_\alpha (= q^*{\restriction}Y_\alpha)$. As before, the latter means that $G^q \cap Y_\alpha = G^{q_0^*} \cap Y_\alpha$ and $(t_b^q, \dot{h}_b^q) = (t_b^{q_0^*}, \dot{h}_b^{q_0^*})$ for $b \in G^q \cap Y_\alpha$. (This is unproblematic because the $\dot{h}_b^{q_0^*}$ are $\mathbb{C}_{(X\cup X_\alpha)\cap X_b}$–names.) $q \leq \bar{q}$ is trivial but, using $\bar{q} \leq \bar{q}_0^*$ and the way q_0^* and q were defined, we also get $q \leq q_0^*$. Therefore

$$q \Vdash \text{“}\dot{h}_\alpha(n) \geq \dot{f}_\alpha(n) \text{ for all } n \geq k\text{”}$$

so that

$$q \Vdash \text{“}\dot{g}_{\omega_2}(n) \geq \dot{f}_\alpha(n) \text{ for all } n \geq k,\text{”}$$

as required. □

□

□

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The Graduate School of Science and Technology, Kobe University, Rokko–dai 1–1, Nada–ku Kobe 657–8501, Japan

E-mail address: brendle@kurt.scitec.kobe-u.ac.jp

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PRODUCTS OF 2 SPACES AND PRODUCTS OF 3 SPACES

YASUSHI HIRATA AND NOBUYUKI KEMOTO

A space is paracompact (subparacompact, metacompact) if for every open cover $\mathcal{U}$, there is a locally finite open (σ-locally finite closed, point finite open) refinement $\mathcal{V}$ of $\mathcal{U}$. A space is submetacompact if for every open cover $\mathcal{U}$, there is a sequence $\{\mathcal{V}_n : n \in \omega\}$ of open refinements of $\mathcal{U}$ such that for each point x, there is $n \in \omega$ such that $\mathcal{V}_n$ is point finite at x. It is well known that paracompactness implies both of subparacompactness and metacompactness, and submetacompactness is a common weakening of subparacompactness and metacompactness. We discuss on these four properties (and its restricted version) of product spaces of ordinal numbers with the usual order topology. The Greek letters α, β, γ... denote ordinals. One of basic results is:

Proposition 1 ([8])**.** *Let A and B be subspaces of α and $X = A \times B$. Then paracompactness, subparacompactness, metacompactness and submetacompactness of X are equivalent.*

Thus, these properties are equivalent for "products of subspaces". But in general, these properties need not be equivalent for "subspaces of products".

Proposition 2 ([7])**.**

(1) *For every subspace X of α^2, metacompactness and submetacompactness of X are equivalent. Thus paracompactness of X implies subparacompactness and subparacompactness of X implies metacompactness.*
(2) *There is a metacompact subspace of $(\omega_2+1)^2$ which is not subparacompact.*
(3) *There is a subparacompact subspace of $(\omega_1+1)^2$ which is not paracompact.*

Now let's consider the restricted versions of these four properties. A space is countably paracompact if for every countable open cover $\mathcal{U}$, there is a locally finite open refinement $\mathcal{V}$ of $\mathcal{U}$. Countable subparacomactness, countable metacompactness and countable submetacompactness are similarly defined. It is well known that countable metacompactness coincides with countable submetacompactness and it is a common weakening of countable paracompactness and countable subparacompactness. But, as is witnessed by the product $\omega_1 \times (\omega_1+1)$, countable paracompactness does not imply countable subparacompactness. The square of Sorgenfrey line is countably subparacompact but not countably paracompact. Thus countable subparacompactness and countable paracompactness are incomparable. Normal (Subnormal) countably metacompact spaces are countably paracompact (countably subparacompact), where a space is normal (subnormal) if every pair of disjoint closed sets are separated by disjoint open sets (G_δ-sets). Of course, normal spaces are subnormal and countably subparacompact spaces are subnormal. But it is well known that countable paracompactness and normality are incomparable. Now we focus on these restricted properties of subspaces of ω_1^n, $n \leq \omega$.

One of basic results about countable paracompactness is:

Proposition 3 ([3]). *Let A and B be subspaces of ω_1 and $X = A \times B$. Then X is countably paracompact iff X is normal iff A or B is non-stationary or $A \cap B$ is stationary.*

Note that for disjoint stationary sets A and B of ω_1, $X = A \times B$ is neither normal nor countably paracompact. It is natural to ask whether normality and coutable paracompactness are equivalent for every subspace of ω_1^2. A consistently affirmative answer is known:

Proposition 4 ([6]). *Assuming $V = L$ or $PMEA$, normality and countable paracompactness are equivalent for every subspace of ω_1^2.*

But, it still remains open whether this is a theorem of ZFC. Note that Proposition 5 bellow shows that normality implies countable paracompactness in ZFC.

For countable metacompactness, we have the following unexpected results.

Proposition 5 ([4]). *All subspaces of ω_1^2 are countably metacompact. Thus normal subspaces of ω_1^2 are countably paracompact.*

Proposition 6 ([5]). *All subspaces of ω_1^n, $n \in \omega$, are countably metacompact, but there is a subspace of ω_1^ω which is not countably metacompact.*

For countable subparacompactness, we also have the following unexpected result.

Proposition 7 ([2]). *All subspaces of ω_1^2 are countably subparacompact.*

So, as in the countably metacompact case, we have no doubt to conjecture that all subspaces of ω_1^n, $n \in \omega$, are countably subparacompact. But unfortunately we obtain:

Theorem 8 ([1]). *There is a subspace of ω_1^3 which is not countably subparacompact.*

The subspace $X = \{\langle \alpha, \beta, \gamma \rangle \in \omega_1^3 : \alpha \leq \beta \leq \gamma\} \setminus \{\langle \alpha, \alpha, \alpha \rangle \in \omega_1^3 : \alpha < \omega_1\}$ of ω_1^3 is the desired one. Indeed, we can show that the disjoint closed subsets $F_0 = \{\langle \alpha, \beta, \gamma \rangle \in X : \alpha = \beta\}$ and $F_1 = \{\langle \alpha, \beta, \gamma \rangle \in X : \beta = \gamma\}$ cannot be separated by disjoint G_δ-sets. So X is not subnormal, and therefore X is a non-countably subparacompact subspace of $\{\langle \alpha, \beta, \gamma \rangle \in \omega_1^3 : \alpha \leq \beta \leq \gamma\}$. But some special subspaces are hereditarily countably subparacompact.

Theorem 9 ([1]). *All subspaces of $\{\langle \alpha, \beta, \gamma \rangle \in \omega_1^3 : \alpha < \beta < \gamma\}$ are countably subparacompact. More generally, all subspaces of $\{x \in \omega_1^n : x(0) < x(1) < \ldots < x(n-1)\}$ are countably subparacompact.*

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(Y. Hirata) Graduate School of Mathematics, University of Tsukuba, Ibaraki 305-8571, Japan
E-mail address, Y. Hirata: yhira@jb3.so-net.ne.jp

(N. Kemoto) Department of Mathematics, Faculty of Education, Oita University, Dannoharu, Oita, 870-1192, Japan
E-mail address, N. Kemoto: nkemoto@cc.oita-u.ac.jp

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HYPERSPACES WITH THE HAUSDORFF METRIC

MASAYUKI KURIHARA, KATSURO SAKAI, AND MASATO YAGUCHI

1. Introduction

This note is a presentation of the results obtained in the paper [8].

Let $X = (X, d)$ be a metric space. The set of all non-empty closed sets in X is denoted by $\mathrm{Cld}(X)$. On the subset $\mathrm{Bdd}(X) \subset \mathrm{Cld}(X)$ consisting of bounded closed sets in X, we can define the *Hausdorff metric* d_H as follows:

$$d_H(A, B) = \max\left\{ \sup_{x\in B} d(x, A),\ \sup_{x\in A} d(x, B) \right\},$$

where $d(x, A) = \inf_{a\in A} d(x, a)$. We denote the metric space $(\mathrm{Bdd}(X), d_H)$ by $\mathrm{Bdd}_H(X)$. On the whole set $\mathrm{Cld}(X)$, we allow $d_H(A, B) = \infty$, but d_H induces the topology of $\mathrm{Cld}(X)$ like a metric does. The space $\mathrm{Cld}(X)$ with this topology is denoted by $\mathrm{Cld}_H(X)$. When X is bounded, $\mathrm{Cld}_H(X) = \mathrm{Bdd}_H(X)$. Even though X is unbounded, $\mathrm{Cld}_H(X)$ is metrizable. Indeed, let $\bar{d}$ be the metric on X defined by $\bar{d}(x, y) = \min\{1, d(x, y)\}$. Then, $\bar{d}_H$ is an admissible metric of $\mathrm{Cld}_H(X)$. It should be noted that each component of $\mathrm{Cld}_H(X)$ is contained in $\mathrm{Bdd}(X)$ or in the complement $\mathrm{Cld}(X) \setminus \mathrm{Bdd}(X)$. Thus, $\mathrm{Bdd}_H(X)$ is a union of components of $\mathrm{Cld}_H(X)$. On each component of $\mathrm{Cld}_H(X)$, d_H is a metric even if it is contained in $\mathrm{Cld}(X) \setminus \mathrm{Bdd}(X)$. Then, we regard every component of $\mathrm{Cld}_H(X)$ as a metric space with d_H.

When X is compact, it is well-known that $\mathrm{Cld}_H(X)$ $(= \mathrm{Bdd}_H(X))$ is an ANR (an AR)[1] if and only if X is locally connected (connected and locally connected) [14]. However, in case X is non-compact, this does not hold. In [8], we construct a metric AR X such that $\mathrm{Cld}_H(X)$ is not an ANR and give a condition on X such that $\mathrm{Cld}_H(X)$ is an ANR. Due to our result, $\mathrm{Cld}_H(X)$ can be an ANR even if X is not locally connected.

2. Main results and a counter-example

Let $X = (X, d)$ be a metric space. For $A \subset X$ and $\gamma > 0$, we denote

$$N(A, \gamma) = \{x \in X \mid d(x, A) < \gamma\} \text{ and}$$
$$\overline{N}(A, \gamma) = \{x \in X \mid d(x, A) \leqslant \gamma\}.$$

When $A = \{a\}$, we write $N(\{a\}, \gamma) = \mathrm{B}(a, \gamma)$ and $\overline{N}(\{a\}, \gamma) = \overline{\mathrm{B}}(a, \gamma)$

In [10], Michael introduced uniform AR's and uniform ANR's. A *uniform ANR* is a metric space X with the property: for an arbitrary metric space $Z = (Z, d)$ containing X isometrically as a closed subset, there exist a uniform neighborhood

[1] An ANR (an AR) means an absolute neighborhood retract (an absolute retract) for metrizable spaces.

U of X in Z (i.e., $U = N(X, \gamma)$ for some $\gamma > 0$) and a retraction $r : U \to X$ which is *uniformly continuous at* X, that is, for each $\varepsilon > 0$, there is some $\delta > 0$ such that if $x \in X$, $z \in U$ and $d(x, z) < \delta$ then $d(x, r(z)) < \varepsilon$. When $U = Z$ in the above, X is called a *uniform AR*. A uniform ANR is a uniform AR if it is homotopically trivial, that is, all the homotopy groups are trivial. In [11], it is shown that a metric space X is a uniform ANR if and only if every metric space Z containing X isometrically as a dense subset is a uniform ANR and X is *homotopy dense* in Z, that is, there exists a homotopy $h : Z \times \mathbf{I} \to Z$ such that $h_0 = \mathrm{id}_Z$ and $h_t(Z) \subset X$ for $t > 0$.

For each $\eta > 0$, a finite sequence $(x_i)_{i=0}^k$ of points in X is called an *η-chain* if $d(x_i, x_{i-1}) < \eta$ for each $i = 1, \dots, k$, where k is the *length* of $(x_i)_{i=0}^k$. The diameter of $(x_i)_{i=0}^k$ means $\operatorname{diam}\{x_i \mid i = 0, 1, \dots, k\}$. When $x_0 = x$ and $x_k = y$, we call $(x_i)_{i=0}^k$ an *η-chain from x to y* and we say that x and y are *connected* by $(x_i)_{i=0}^k$. It is said that X is *C-connected* (*connected in the sense of Cantor*) if each pair of points in X are connected by an η-chain in X for any $\eta > 0$. Now, we say that X is *uniformly locally C^*-connected* if for each $\varepsilon > 0$ there exists $\delta > 0$ with the following property:

(ulC^*) For each $\eta > 0$, there is $k \in \mathbb{N}$ such that each pair of δ-close points of X are connected by an η-chain with the length $\leqslant k$ and the diameter $< \varepsilon$.

It is easy to see that a metric space is uniformly locally C^*-connected if it is uniformly homeomorphic to a uniformly locally C^*-connected metric space.

A collection $\mathcal{A}$ of subsets of X is said to be *uniformly discrete* if there exists some $\delta > 0$ such that the δ-neighborhood $\mathrm{B}(x, \delta)$ of each $x \in X$ meets at most one member of $\mathcal{A}$, that is,

$$\inf\{\operatorname{dist}(A, A') \mid A \neq A' \in \mathcal{A}\} > 0,$$

where $\operatorname{dist}(A, A') = \inf\{d(x, x') \mid x \in A,\ x' \in A'\}$. The following is the main result in [8]:

Theorem. *For every uniformly locally C^*-connected metric space X, the collection of all components of* $\mathrm{Cld}_H(X)$ *is uniformly discrete and each component of the space* $\mathrm{Cld}_H(X)$ *is a uniform AR, hence* $\mathrm{Cld}_H(X)$ *is an ANR and* $\mathrm{Bdd}_H(X)$ *is a uniform ANR.*

By the main Theorem above, for every dense subset X of a convex set in a normed linear space, each component of $\mathrm{Cld}_H(X)$ is a uniform AR and $\mathrm{Bdd}_H(X)$ is a uniform ANR. Recently, Constantini and Kubiś [4] showed that $\mathrm{Bdd}_H(X)$ is an AR if X is almost convex, where X is *almost convex* if for each $x, y \in X$ and for each $s, t > 0$ such that $d(x, y) < s + t$, there exists $z \in X$ with $d(x, z) < s$ and $d(y, z) < t$. This result follows from the main Theorem because every almost convex metric space is uniformly locally C^*-connected.

Here, we construct a metric AR X such that $\mathrm{Cld}_H(X)$ is not an ANR.

Example 1. As a subspace of Euclidean space $\mathbb{R}^3$, let $X = \bigcup_{n \in \mathbb{N} \cup \{0\}} X_n$, where

$$X_0 = \{(x, xz, z) \in \mathbb{R}^3 \mid x \geqslant 1,\ z \in \mathbf{I}\} \text{ and}$$

$$X_n = \{(x, y, 1/n) \in \mathbb{R}^3 \mid x \geqslant 1,\ 0 \leqslant y \leqslant x/n\} \text{ for } n \in \mathbb{N}.$$

Then, X is an AR. In fact, X is homeomorphic to ($\approx$) the following space:

$$\left(\mathbf{I} \times \{0\} \cup \bigcup_{n \in \mathbb{N}} \{1/n\} \times [0, 1/n]\right) \times [1, \infty).$$

Moreover, it can be proved that $\mathrm{Cld}_H(X)$ is not locally path-connected at $A = \mathbb{N} \times \{0\} \times \{0\}$, hence is not an ANR.

In the above example, X is not a uniform ANR. The following is unknown:

Problem 1. For every uniform ANR X, is $\mathrm{Cld}_H(X)$ an ANR?

3. Lawson semilattices which are uniform ANR's

A *topological semilattice* is a topological space S equipped with a continuous operator $\vee : S \times S \to S$ which is reflexive, commutative and associative (i.e., $x \vee x = x$, $x \vee y = y \vee x$, $(x \vee y) \vee z = x \vee (y \vee z)$). A topological semilattice S is called a *Lawson semilattice* if S admits an open basis consisting of subsemilattices [9]. It is known that a metrizable Lawson semilattice is k-aspherical for each $k > 0$ ([4, Proposition 2.3]).

In [1], it is shown that a metrizable Lawson semilattice is an ANR (resp. an AR) if and only if it is locally path-connected (resp. connected and locally path-connected). Here, we consider the condition that a metric Lawson semilattice is a uniform ANR.

It is said that a metric space X is *uniformly locally contractible* if for each $\varepsilon > 0$, there exist $\delta > 0$ such that the δ-ball $\mathrm{B}(x, \delta)$ at each $x \in X$ is contractible in the ε-ball $\mathrm{B}(x, \varepsilon)$. Every uniform ANR is uniformly locally contractible by [10, Proposition 1.5 and Theorem 1.6]. And, as is easily observed, every uniformly locally contractible metric space is uniformly locally k-connected for all $k \geqslant 0$.

We have the following characterization:

Theorem 3.1. *Let $L = (L, d, \vee)$ be a metric Lawson semilattice such that*

$$d(x \vee x', y \vee y') \leqslant \max\{d(x, y), d(x', y')\} \text{ for each } x, x', y, y' \in L.$$

Then, the following are equivalent:

(a) *the collection of all components of L is uniformly discrete in L and each component of L is a uniform AR;*
(b) *L is a uniform ANR;*
(c) *L is uniformly locally contractible;*
(d) *L is uniformly locally path-connected.*

4. The uniform local C^*-connectedness

For two metric spaces $X = (X, d_X)$ and $Y = (Y, d_Y)$, let $C(X, Y)$ be the collection of all continuous functions from X to Y. It is said that $\mathcal{F} \subset C(X, Y)$ is *uniformly equi-continuous* if for each $\varepsilon > 0$, there is $\delta > 0$ such that $d_Y(f(x), f(x')) < \varepsilon$ for each $f \in \mathcal{F}$ and $x, x' \in X$ with $d_X(x, x') < \delta$. We can characterize the uniform locall C^*-connectedness as follows:

Theorem 4.1. *Let D be a countable dense subset of the unit interval* $\mathbf{I}$ *with the usual metric and $0, 1 \in D$. Then, a metric space $X = (X, d)$ is uniformly locally C^*-connected if and only if for each $\varepsilon > 0$, there exist $\delta > 0$ and $\mathcal{F} \subset C(D, X)$ satisfying the following:*

(i) *$\mathcal{F}$ is uniformly equi-continuous,*
(ii) $\mathrm{diam}\, f(D) < \varepsilon$ *for every $f \in \mathcal{F}$,*
(iii) *for each δ-close $x, y \in X$, there is $f \in \mathcal{F}$ with $f(0) = x$ and $f(1) = y$.*

For a complete metric space X and a dense subset $D \subset \mathbf{I}$, since every uniformly continuous map $f : D \to X$ extends over $\mathbf{I}$, the following follows from Theorem 4.1:

Corollary 4.2. *Every uniformly locally C^*-connected complete metric space is uniformly locally path-connected.*

It is known that $\mathrm{Cld}_H(X) = (\mathrm{Cld}_H(X), \cup)$ is a Lawson semilattice satisfying the following condition:

$$d_H(A \cup A', B \cup B') \leqslant \max\{d_H(A, B),\ d_H(A', B')\} \quad \text{for each } A, A', B, B' \in \mathrm{Cld}_H(X).$$

Refer to [4, Proposition 2.4] (cf. the proof of [1, Fact 4]). The following can be proved:

Theorem 4.3. *For every uniformly locally C^*-connected metric space X, the space $\mathrm{Cld}_H(X)$ is uniformly locally path-connected.*

Combining this result with Theorem 3.4, we can obtain the main Theorem.

5. The uniformly local almost convexity

A metric space $X = (X, d)$ is *locally almost convex* if each $x \in X$ has a neighborhood U such that

(lac) for each $y, z \in U$ and for each $s, t > 0$ with $s + t > d(y, z)$, there is $w \in X$ such that $d(y, w) < s$ and $d(w, z) < t$.

It is said that X is *uniformly locally almost convex* if there is some $\delta > 0$ such that

(ulac) for each $x, y \in X$ with $d(x, y) < \delta$ and for each $s, t > 0$ with $s+t > d(x, y)$, there is some $z \in X$ such that $d(x, z) < s$ and $d(y, z) < t$.

Note that an almost convex metric space is uniformly locally almost convex. Obviously, a uniformly locally almost convex metric space is locally almost convex, but the converse does not hold as the example below:

Example 2. The following subspace of $\mathbb{R}^2$ inherited the Euclidean metric is clearly locally almost convex:

$$X = \mathbb{N} \times [0, \infty) \setminus \bigcup_{n \in \mathbb{N}} \{n\} \times (0, 2^{-n}).$$

However, X is not uniformly locally almost convex.

The following are characterizations of uniformly locally almost convexity and almost convexity:

Theorem 5.1. *For a metric space $X = (X, d)$, the following are equivalent:*

(a) *X is uniformly locally almost convex;*
(b) *there exists some $\delta > 0$ such that for each $x, y \in X$ with $d(x, y) < \delta$ and for each $\varepsilon > 0$, there is some $z \in X$ with $d(x, z), d(y, z) < \frac{1}{2} d(x, y) + \varepsilon$;*
(c) *there exists $\delta > 0$ such that for each $0 < \lambda < \delta$ and for each $x, y \in X$ with $d(x, y) < \lambda$, there exist a λ-Lipschitz map $f : D \to X$ with $f(0) = x$ and $f(1) = y$;*
(d) *there is $\delta > 0$ such that $N(N(A, s), t) = N(A, s + t)$ for each $A \subset X$ and for each $s, t > 0$ with $s + t \leqslant \delta$.*

Theorem 5.2. *For a metric space $X = (X, d)$, the following are equivalent:*

(a) *X is almost convex;*
(b) *for each $x, y \in X$ and for each $\varepsilon > 0$, there is some $z \in X$ with $d(x, z)$, $d(y, z) < \frac{1}{2}d(x, y) + \varepsilon$;*
(c) *for each $\lambda > 0$ and for each $x, y \in X$ with $d(x, y) < \lambda$, there exist a λ-Lipschitz map $f : D \to X$ with $f(0) = x$ and $f(1) = y$;*
(d) *$N(N(A, s), t) = N(A, s + t)$ for each $A \subset X$ and for each $s, t > 0$.*

As is easily observed, every uniformly locally almost convex metric space X is uniformly locally C^*-connected. Hence, we have the following:

Corollary 5.3. *For every uniformly locally almost convex metric space X, each component of $\mathrm{Cld}_H(X)$ is a uniform AR and $\mathrm{Bdd}_H(X)$ is a uniform ANR.*

One should note that the unit circle $\mathbf{S}^1 \subset \mathbb{R}^2$ with the Euclidean metric is uniformly locally C^*-connected but not uniformly locally almost convex.

Recall a metric space $X = (X, d)$ (or a metric d) is *convex* if for each $x, y \in X$, there is some $z \in X$ with $d(x, z) = d(y, z) = d(x, y)/2$. A complete metric space X is convex if and only if for each $x, y \in X$, there is a map $f : [0, d(x, y)] \to X$ with $d(x, f(t)) = t$. As is easily observed, every almost convex compact metric space is convex.

Problem 2. Does there exist an almost convex *complete* metric space which is not convex?

It is well-know that a Peano continuum[2] has an admissible convex metric (cf. [3]). It is said that X is *locally convex* if each $x \in X$ has a neighborhood U which is convex. Moreover, X is *uniformly locally convex* if there is some $\delta > 0$ such that for each $x, y \in X$ with $d(x, y) < \delta$ there is $z \in X$ with $d(x, z) = d(y, z) = d(x, y)/2$.

Problem 3. Does a locally connected metric space possess an admissible metric that is locally convex, (uniformly) locally almost convex, or uniformly locally D-connected, uniformly locally C^*-connected?

6. The uniformly local C-connectedness

In this section, we show that the uniformly local C^*-connectedness is a stronger condition than the uniformly local version of C-connectedness. It is said that X is *uniformly locally C-connected* if for each $\varepsilon > 0$ there exists $\delta > 0$ with the following property:

(ul$C$) for each $\eta > 0$ and each $x, y \in X$ with $d(x, y) < \delta$, there is an η-chain $(x_n)_{n=0}^k$ in X from x to y with the diameter $< \varepsilon$.

Proposition 6.1. *If $\mathrm{Bdd}_H(X)$ is uniformly locally path-connected, then X is uniformly locally C-connected.*

Every uniformly locally path-connected metric space is uniformly locally C-connected, but the converse does not hold as the space $\mathbb{Q}$ of rationals. Moreover, every uniformly locally C^*-connected metric space is uniformly locally C-connected, but the converse does not hold as the following example:

[2]A connected compact metrizable space is called a *continuum* and a locally connected continuum is called a *Peano continuum.*

Example 3. For each $n \in \mathbb{N}$, let e_n be the unit vector in $\mathbb{R}^{\mathbb{N}}$ defined $e_n(i) = 0$ if $i \neq n$ and $e_n(n) = 1$. We define a metric space $X = (X, d)$ as follows:

$$X = \bigcup_{n\in\mathbb{N}} \mathbb{R}e_n \subset \mathbb{R}^{\mathbb{N}},\ d(x,y) = \sum_{n\in\mathbb{N}} \min\{2^{-n}, |x(n) - y(n)|\}.$$

Then, X is uniformly locally C-connected but it is not uniformly locally C^*-connected.

Remark 1. In Example 3 above, $\mathrm{Cld}_H(X)$ is locally path-connected, hence so is $\mathrm{Bdd}_H(X)$ and they would be ANR's.

As saw in the above, in order that $\mathrm{Cld}_H(X)$ (or $\mathrm{Bdd}_H(X)$) is an ANR, it is not necessary that X is uniformly locally C^*-connected. The following problems are open:

Problem 4. When $\mathrm{Bdd}_H(X)$ is uniformly locally path-connected (hence it is a uniform ANR), is X uniformly locally C^*-connected?

Problem 5. Does the converse of Proposition 6.1 above hold? Or, for each uniformly locally C-connected metric space X, is $\mathrm{Cld}_H(X)$ (uniformly) locally path-connected?

It is easy to see that every uniformly locally path-connected metric space is uniformly locally C-connected.

Problem 6. For each uniformly locally path-connected metric space X, is $\mathrm{Cld}_H(X)$ (or $\mathrm{Bdd}_H(X)$) (uniformly) locally path-connected?

Theorem 6.2. *Let (X, d) be a uniformly locally C-connected metric space. Then, each $x \in X$ has a neighborhood basis consisting of C-connected open neighborhood of x.*

It is known that every compact C-connected set is connected. By using this fact, we can prove the following lemma.

Lemma 6.3. *Every locally compact uniformly locally C-connected metric space X is locally connected.*

Theorem 6.4. *Let $X = (X, d)$ be a totally bounded uniformly locally C-connected metric space. Then, X has a uniformly locally almost convex metric which is uniformly equivalent to d.*

By Theorem 6.4 above and Corollary 5.3, we have the following:

Corollary 6.5. *For every totally bounded uniformly locally C-connected metric space X, the space $\mathrm{Cld}_H(X)$ $(= \mathrm{Bdd}_H(X))$ is a uniform ANR.*

Problem 7. In Theorem 6.4 and Corollary 6.5, is the total boundedness essential?

Problem 8. For *complete metric spaces*, does the C-connectedness imply the connectedness?

7. Further problems and related results

The following proposition is shown in [7], which shows the complexity of the space $\mathrm{Cld}_H(X)$.

Proposition 7.1. *The space* $\mathrm{Cld}_H(\mathbb{R}^n)$ *has uncountably many components. Moreover, the space of all compact sets* $\mathrm{Comp}_H(\mathbb{R}^n)$ *is one of them and all but this component are non-separable.*

The following is known.

Theorem 7.2. *If a metric space* $X = (X, d)$ *is complete, then so is every component of* $\mathrm{Cld}_H(X)$*. Consequently,* $\mathrm{Bdd}_H(X)$ *is complete.*

By the main Theorem and Theorem 7.2, for an arbitrary Banach space X, every component of $\mathrm{Cld}_H(X)$ is a complete metric AR.

Problem 9. For a Banach space (or a Hilbert space) X, is every component of $\mathrm{Cld}_H(X)$ homeomorphic to a Hilbert space?

Even if X is Euclidean space $\mathbb{R}^n$, the above is unknown, that is,

Problem 10. Is each non-separable component of $\mathrm{Cld}_H(\mathbb{R}^n)$ homeomorphic to a Hilbert space?

In relation to above problems, some results with different topologies have been obtained in [1], [6], [12] and [13]. For topologies on hyperspaces, we refer to the book [2]. Let $Q = \prod_{i\in\mathbb{N}}[-2^{-i}, 2^{-i}]$ be the Hilbert cube and $B(Q) = Q \setminus \prod_{i\in\mathbb{N}}(-2^{-i}, 2^{-i})$ be the pseudo-boundary of Q. By $\ell_2(\tau)$, we denote the Hilbert space with weight τ. Let ℓ_2^f and Q_f be the subspaces of the separable Hilbert space ℓ_2 and Q respectively, defined as follows:

$$\ell_2^f = \{(x_i)_{i\in\mathbb{N}} \in \ell_2 \mid x_i = 0 \text{ except for finitely many } i \in \mathbb{N}\}.$$
$$Q_f = \{(x_i)_{i\in\mathbb{N}} \in Q \mid x_i = 0 \text{ except for finitely many } i \in \mathbb{N}\}.$$

Theorem 7.3. [13] *For a Hausdorff space* X*, the hyperspace* $\mathrm{Cld}_F(X)$ *with the Fell topology is homeomorphic to* $Q \setminus \{0\}$ *if and only if* X *is locally compact, locally connected, separable metrizable and has no compact components, whence* $\mathrm{Comp}_F(X) \approx B(Q)$*. In case* X *is strongly countable-dimensional,* $\mathrm{Fin}_F(X) \approx Q_f$*.*

Theorem 7.4. [1], [12] *For every infinite-dimensional Banach space* X *with weight* τ*, the hyperspace* $\mathrm{Cld}_{AW}(X)$ *with Attouch-Wets topology is homeomorphic to* $\ell_2(2^\tau)$*,*

$$\mathrm{Fin}_{AW}(X) \approx \mathrm{Comp}_{AW}(X) \approx \ell_2(\tau) \times \ell_2^f \quad \textit{and}$$
$$\mathrm{Bdd}_{AW}(X) \approx \ell_2(2^\tau) \times \ell_2^f.$$

Theorem 7.5. [6] *For every infinite-dimensional separable Banach space* X*, the hyperspace* $\mathrm{Cld}_W(X)$ *with the Wijsman topology is homeomorphic to* ℓ_2 *and*

$$\mathrm{Fin}_W(X) \approx \mathrm{Bdd}_W(X) \approx \ell_2 \times \ell_2^f.$$

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(M. Kurihara) Institute of Mathematics, University of Tsukuba, Tsukuba, 305-8571 JAPAN
E-mail address: kurihara@first.tsukuba.ac.jp

(K. Sakai) Institute of Mathematics, University of Tsukuba, Tsukuba, 305-8571 JAPAN
E-mail address: sakaiktr@sakura.cc.tsukuba.ac.jp

(M. Yaguchi) Institute of Mathematics, University of Tsukuba, Tsukuba, 305-8571 JAPAN
E-mail address: masato@math.tsukuba.ac.jp

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APPROXIMATE RESOLUTIONS AND AN APPLICATION TO HAUSDORFF DIMENSION

TAKAHISA MIYATA AND TADASHI WATANABE

Abstract. In this paper, we present a new approach using normal sequences and approximate inverse systems to study Hausdorff dimension for compact metrizable spaces.

This short article summarizes the main results from the paper [8].

1. Preliminary

For each subset F of $\mathbb{R}^m$ and for each $s \geq 0$, the *s-dimensional Hausdorff measure* of F is defined as $\mathrm{H}^s(F) = \lim\limits_{\delta \to 0} \mathrm{H}^s_\delta(F)$ where for each $\delta > 0$,

$$\mathrm{H}^s_\delta(F) = \inf \sum_{i=1}^{\infty} |U_i|^s$$

where the infimum is taken over all open balls U_i with radius at most δ such that $F \subseteq \bigcup\limits_{i=1}^{\infty} U_i$. Here $|U_i|$ denotes the diameter of the set U_i. The *Hausdorff dimension* of F is defined as $\dim_H F = \sup\{s : \mathrm{H}^s(F) = \infty\}$ $(= \inf\{s : \mathrm{H}^s(F) = 0\})$ [2]. The present paper concerns Hausdorff dimension for non-Euclidean spaces. More precisely, we develop a systematic approach using normal sequences and approximate inverse systems to study Hausdorff dimension for compact metrizable spaces.

Throughout the paper, *all spaces are assumed to be metrizable, and maps mean continuous maps.*

For any space X, let $\mathrm{Cov}(X)$ denote the family of all open coverings of X. For any $\mathfrak{U}, \mathfrak{V} \in \mathrm{Cov}(X)$, $\mathfrak{U}$ is a *refinement* of $\mathfrak{V}$, in notation, $\mathfrak{U} < \mathfrak{V}$, if for each $U \in \mathfrak{U}$ there is $V \in \mathfrak{V}$ such that $U \subseteq V$. For any subset A of X and $\mathfrak{U} \in \mathrm{Cov}(X)$, let $\mathrm{st}(A, \mathfrak{U}) = \cup\{U \in \mathfrak{U} : U \cap A \neq \emptyset\}$ and $\mathfrak{U}|A = \{U \cap A : U \in \mathfrak{U}\}$. If $A = \{x\}$, we write $\mathrm{st}(x, \mathfrak{U})$ for $\mathrm{st}(\{x\}, \mathfrak{U})$. For each $\mathfrak{U} \in \mathrm{Cov}(X)$, let $\mathrm{st}\,\mathfrak{U} = \{\mathrm{st}(U, \mathfrak{U}) : U \in \mathfrak{U}\}$. Let $\mathrm{st}^1 \mathfrak{U} = \mathrm{st}\,\mathfrak{U}$ and $\mathrm{st}^{n+1} \mathfrak{U} = \mathrm{st}(\mathrm{st}^n \mathfrak{U})$ for each $n \in \mathbb{N}$. For any metric space (X, d) and $r > 0$, let $\mathrm{U}_\mathrm{d}(x, r) = \{y \in X : \mathrm{d}(x, y) < r\}$, and for each subset A of X, let $|A|$ denote the diameter of A. For any $\mathfrak{U} \in \mathrm{Cov}(X)$, two points $x, x' \in X$ are $\mathfrak{U}$-*near*, denoted $(x, x') < \mathfrak{U}$, provided $x, x' \in U$ for some $U \in \mathfrak{U}$. For any $\mathfrak{V} \in \mathrm{Cov}(Y)$, two maps $f, g : X \to Y$ between spaces are $\mathfrak{V}$-*near*, denoted $(f, g) < \mathfrak{V}$, provided $(f(x), g(x)) < \mathfrak{V}$ for each $x \in X$. For each $\mathfrak{U} \in \mathrm{Cov}(X)$ and $\mathfrak{V} \in \mathrm{Cov}(Y)$, let

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$f(\mathfrak{U}) = \{f(U) : U \in \mathfrak{U}\}$ and $f^{-1}(\mathfrak{V}) = \{f^{-1}(V) : V \in \mathfrak{V}\}$. Let I denote the closed interval $[0, 1]$, and let $\mathbb{N}$ denote the set of all positive integers.

Metrics induced by normal sequences. A family $\mathbb{U} = \{\mathfrak{U}_i : i \in \mathbb{N}\}$ of open coverings on a space X is said to be a *normal sequence* provided $\operatorname{st}\mathfrak{U}_{i+1} < \mathfrak{U}_i$ for each i. Let $\Sigma\mathbb{U}$ denote the normal sequence $\{\mathfrak{V}_i : \mathfrak{V}_i = \mathfrak{U}_{i+1}, i \in \mathbb{N}\}$ and $\operatorname{st}\mathbb{U}$ the normal sequence $\{\operatorname{st}\mathfrak{U}_i : i \in \mathbb{N}\}$. For any normal sequences $\mathbb{U} = \{\mathfrak{U}_i\}$ and $\mathbb{V} = \{\mathfrak{V}_i\}$, we write $\mathbb{U} < \mathbb{V}$ provided $\mathfrak{U}_i < \mathfrak{V}_i$ for each i. For each map $f : X \to Y$ and for each normal sequence $\mathbb{V} = \{V_i\}$, let $f^{-1}(\mathbb{V}) = \{f^{-1}(\mathfrak{V}_i)\}$. For each subspace A of X, let $\mathbb{U}|A$ denote the normal sequence $\{\mathfrak{U}_i|A\}$ on A.

Following the approach by Alexandroff and Urysohn, given a space X and a normal sequence $\mathbb{U} = \{\mathfrak{U}_i\}$ on X, we define the metric $\mathrm{d}_{\mathbb{U}}$ on X as follows (for more details, see [6]):

$$\mathrm{d}_{\mathbb{U}}(x, x') = \inf\{\mathrm{D}_{\mathbb{U}}(x, x_1) + \mathrm{D}_{\mathbb{U}}(x_1, x_2) + \cdots + \mathrm{D}_{\mathbb{U}}(x_n, x')\}$$

where the infimum is taken over all points $x_1, x_2, ..., x_n$ in X, and

$$\mathrm{D}_{\mathbb{U}}(x, x') = \begin{cases} 9 & \text{if } (x, x') \not< \mathfrak{U}_1; \\ \frac{1}{3^{i-2}} & \text{if } (x, x') < \mathfrak{U}_i \text{ but } (x, x') \not< \mathfrak{U}_{i+1}\ ; \\ 0 & \text{if } (x, x') < \mathfrak{U}_i \text{ for all } i \in \mathbb{N}\ , \end{cases}$$

Then the metric $\mathrm{d}_{\mathbb{U}}$ has the property

$$\operatorname{st}(x, \mathfrak{U}_{i+3}) \subseteq \mathrm{U}_{\mathrm{d}_{\mathbb{U}}}(x, \frac{1}{3^i}) \subseteq \operatorname{st}(x, \mathfrak{U}_i) \text{ for each } x \in X \text{ and } i.$$

In particular, if $\mathbb{U} = \{\mathfrak{U}_i\}$ is the normal sequence on a metric space (X, d) such that $\mathfrak{U}_i = \{\mathrm{U}_{\mathrm{d}}(x, \frac{1}{3^i}) : x \in X\}$, then the metric $\mathrm{d}_{\mathbb{U}}$ induces the uniformity which is isomorphic to that induced by the metric d. Moreover, if X is a convex subset of a linear toplogical space, then $\mathrm{d}_{\mathbb{U}}$ is isometric to the original metric.

Proposition 1. *Let* $\mathbb{U} = \{\mathfrak{U}_i\}$ *and* $\mathbb{V} = \{\mathfrak{V}_i\}$ *be normal sequences on* X*. Then for all* $x, x' \in X$*,*

(1) *if* $\mathbb{U} < \mathbb{V}$*, then* $\mathrm{d}_{\mathbb{U}}(x, x') \geq \mathrm{d}_{\mathbb{V}}(x, x')$,
(2) $\mathrm{d}_{\Sigma\mathbb{U}}(x, x') = 3\,\mathrm{d}_{\mathbb{U}}(x, x')$,
(3) $\mathrm{d}_{\operatorname{st}\mathbb{U}}(x, x') \leq \mathrm{d}_{\mathbb{U}}(x, x') \leq 3\,\mathrm{d}_{\operatorname{st}\mathbb{U}}(x, x')$.

Approximate sequences and resolutions. An inverse sequence $(X_i, p_{i,i+1})$ consists of spaces X_i, called *coordinate spaces*, and maps $p_{i,i+1} : X_{i+1} \to X_i$. We write p_{ij} for the composite $p_{i,i+1}p_{i+1,i+2} \cdots p_{j-1,j}$ if $i < j$, and let $p_{ii} = 1_{X_i}$, and call maps p_{ij} *bonding maps*. An *approximate inverse sequence* (*approximate sequence*, in short) $\boldsymbol{X} = (X_i, \mathfrak{U}_i, p_{i,i+1})$ consists of an inverse sequence $(X_i, p_{i,i+1})$ and $\mathfrak{U}_i \in \operatorname{Cov}(X_i)$ and must satisfy the following condition:

(AI): For each $i \in \mathbb{N}$ and $\mathfrak{U} \in \operatorname{Cov}(X_i)$, there exists $i' > i$ such that $\mathfrak{U}_{i''} < p_{ii''}^{-1}\mathfrak{U}$ for $i'' > i'$.

An *approximate map* $\boldsymbol{p} = (p_i) : X \to \boldsymbol{X}$ of a compact space X into an approximate sequence $\boldsymbol{X} = (X_i, \mathfrak{U}_i, p_{i,i+1})$ consists of maps $p_i : X \to X_i$ for $i \in \mathbb{N}$, called *projection maps*, such that $p_i = p_{ij}p_j$ for $i < j$, and it is an *approximate resolution* of X if it satisfies the following two conditions:

(R1): For each ANR P, $\mathfrak{V} \in \operatorname{Cov}(P)$ and map $f : X \to P$, there exist $i \in \mathbb{N}$ and a map $g : X_i \to P$ such that $(gp_i, f) < \mathfrak{V}$, and

(R2): For each ANR P and $\mathfrak{V} \in \mathrm{Cov}(P)$, there exists $\mathfrak{V}' \in \mathrm{Cov}(P)$ such that whenever $i \in \mathbb{N}$ and $g, g' : X_i \to P$ are maps with $(gp_i, g'p_i) < \mathfrak{V}'$, then $(gp_{ii'}, g'p_{ii'}) < \mathfrak{V}$ for some $i' > i$.

The following is a useful characterization:

Theorem 2. ([5, Theorem 2.8]) *An approximate map $\boldsymbol{p} = (p_i) : X \to \boldsymbol{X} = (X_i, \mathfrak{U}_i, p_{i,i+1})$ is an approximate resolution of X if and only if it satisfies the following two conditions:*

(B1): *For each $\mathfrak{U} \in \mathrm{Cov}(X)$, there exists $i_0 \in \mathbb{N}$ such that $p_i^{-1}\mathfrak{U}_i < \mathfrak{U}$ for $i > i_0$, and*

(B2): *For each $i \in \mathbb{N}$ and $\mathfrak{U} \in \mathrm{Cov}(X_i)$, there exists $i_0 > i$ such that $p_{ii'}(X_{i'}) \subseteq \mathrm{st}(p_i(X), \mathfrak{U})$ for $i' > i_0$.*

An approximate resolution $\boldsymbol{p} = (p_i) : X \to \boldsymbol{X}$ is said to be *normal* if the family $\mathbb{U} = \{p_i^{-1}(\mathfrak{U}_i)\}$ is a normal sequence. Then each normal approximate resolution $\boldsymbol{p}$ induces a metric $\mathrm{d}_{\mathbb{U}}$, which will be denoted by $\mathrm{d}_{\boldsymbol{p}}$.

Theorem 3 ([5]). *Every compact space X admits a normal approximate resolution $\boldsymbol{p} = (p_i) : X \to \boldsymbol{X} = (X_i, \mathfrak{U}_i, p_{i,i+1})$ such that all coordinate spaces X_i are finite polyhedra.*

Throughout the paper, *every normal approximate resolution is assumed to have the property of Theorem 3.* Our notions of approximate systems and approximate resolutions are the commutative versions of the corresponding notions in [4] and [5].

2. Hausdorff dimension

Let X be a σ-compact space with a normal sequence $\mathbb{U} = \{U_i\}$. For each $s \geq 0$, for each subset F of X and for each $i \in \mathbb{N}$, we define

$$\mathrm{H}^s_{\mathbb{U},i}(F) = \inf\left\{\sum_{k=1}^{\infty}\left(\frac{1}{3^{i_k}}\right)^s : F \subseteq \bigcup_{k=1}^{\infty} U_{i_k}, U_{i_k} \in \mathfrak{U}_{i_k}, i \leq i_k\right\},$$

and

$$\mathrm{H}^s_{\mathbb{U}}(F) = \lim_{i\to\infty} \mathrm{H}^s_{\mathbb{U},i}(F).$$

Then $\mathrm{H}^t_{\mathbb{U},i}(F) \leq \left(\frac{1}{3^i}\right)^{t-s} \mathrm{H}^s_{\mathbb{U},i}(F)$ for $s < t$ and for all i, and hence if $\mathrm{H}^s_{\mathbb{U}}(F) < \infty$, then $\mathrm{H}^t_{\mathbb{U}}(F) = 0$ for $t > s$. Thus we can define the *Hausdorff dimension* $\dim^{\mathbb{U}}_H F$ *of F with respect to* $\mathbb{U}$ by $\dim^{\mathbb{U}}_H F = \inf\{s : \mathrm{H}^s_{\mathbb{U}}(F) = 0\}$ $(= \sup\{s : \mathrm{H}^s_{\mathbb{U}}(F) = \infty\})$. We can prove

Theorem 4. $\mathrm{H}^s_{\mathbb{U}}$ *is a metric outer measure on X with respect to the metric* $\mathrm{d}_{\mathbb{U}}$.

Hence $\mathrm{H}^s_{\mathbb{U}}$ defines a measure on the Borel subsets of X, which we call the *s-dimensional Hausdorff measure with respect to* $\mathbb{U}$ (or *s-dimensional Hausdorff $\mathbb{U}$-measure*) on X.

Our Hausdorff dimension coincides with the usual Hausdorff dimension for Euclidean space with a particular normal sequence.

Theorem 5. *Let $\mathbb{B} = \{\mathfrak{B}_i\}$ be the normal sequence on $\mathbb{R}^n$ which consists of the open coverings $\mathfrak{B}_i$ by open balls with radius $\frac{1}{3^i}$. Then for any subset F of $\mathbb{R}^n$ and for each i and s, $\mathrm{H}^s_{\mathbb{B},i}(F) = 2^s\,\mathrm{H}^s_{\frac{1}{3^i}}(F)$, and hence $\dim^{\mathbb{B}}_H F = \dim_H F$.*

We have the following properties:

Theorem 6. *Let* $\mathbb{U} = \{\mathfrak{U}_i\}$ *and* $\mathbb{V} = \{\mathfrak{V}_i\}$ *be normal sequences on* X*, and let* F *be any subset of* X*.*

(1) *If* $\mathbb{V} < \mathbb{U}$*, then* $\mathrm{H}^s_{\mathbb{U}}(F) \le \mathrm{H}^s_{\mathbb{V}}(F)$*, and hence* $\dim_H^{\mathbb{U}} F \le \dim_H^{\mathbb{V}} F$*.*
(2) $\mathrm{H}^s_{\Sigma\mathbb{U}}(F) = 3^s\,\mathrm{H}^s_{\mathbb{U}}(F)$*, and hence* $\dim_H^{\Sigma\mathbb{U}} F = \dim_H^{\mathbb{U}} F$*.*
(3) $\mathrm{H}^s_{\mathrm{st}\,\mathbb{U}}(F) \le \mathrm{H}^s_{\mathbb{U}}(F) \le 3^s\,\mathrm{H}^s_{\mathrm{st}\,\mathbb{U}}(F)$*, and hence* $\dim_H^{\mathrm{st}\,\mathbb{U}} F = \dim_H^{\mathbb{U}} F$*.*
(4) (Subset theorem) *If* $F_1 \subseteq F_2 \subseteq X$*, then* $\mathrm{H}^s_{\mathbb{U}}(F_1) \le \mathrm{H}^s_{\mathbb{U}}(F_2)$ *and hence* $\dim_H^{\mathbb{U}} F_1 \le \dim_H^{\mathbb{U}} F_2$*.*
(5) (Sum theorem) $\dim_H^{\mathbb{U}}(F_1 \cup F_2) = \max\{\dim_H^{\mathbb{U}} F_1, \dim_H^{\mathbb{U}} F_2\}$ *for any subsets* F_1, F_2 *of* X*.*

For any spaces X and Y with normal sequences $\mathbb{U} = \{U_i\}$ and $\mathbb{V} = \{V_i\}$, respectively, a map $f : X \to Y$ is called a $(\mathbb{U},\mathbb{V})$-*Lipschitz map* provided there exists a constant $\alpha > 0$ such that

$$\mathrm{d}_{\mathbb{V}}(f(x), f(x')) \le \alpha\, \mathrm{d}_{\mathbb{U}}(x, x') \text{ for } x, x' \in X,$$

and a $(\mathbb{U},\mathbb{V})$-*bi-Lipschitz map* provided there exist constants $\alpha_1, \alpha_2 > 0$ such that

$$\alpha_1\, \mathrm{d}_{\mathbb{U}}(x, x') \le \mathrm{d}_{\mathbb{V}}(f(x), f(x')) \le \alpha_2\, \mathrm{d}_{\mathbb{U}}(x, x') \text{ for } x, x' \in X.$$

Lipschitz maps and bi-Lipschitz maps are characterized in terms of normal sequences as follows:

Theorem 7. ([6, §.5, §.7] and [7, §.3]) *Consider the following conditions:*

$\mathbf{(L)}_k$**:** $\mathrm{d}_{\mathbb{V}}(f(x), f(x')) \le 3^k\, \mathrm{d}_{\mathbb{U}}(x, x')$ *for* $x, x' \in X$*,*
$\mathbf{(L)}^k$**:** $\mathrm{d}_{\mathbb{U}}(x, x') \le 3^k\, \mathrm{d}_{\mathbb{V}}(f(x), f(x'))$ *for* $x, x' \in X$*,*
$\mathbf{(N)}_{m,n}$**:** $\Sigma^m\mathbb{U} < f^{-1}(\Sigma^n\mathbb{V})$*,*
$\mathbf{(N)}^{m,n}$**:** $f^{-1}(\Sigma^m\mathbb{V}) < \Sigma^n\mathbb{U}$*.*

Then for $m, n \ge 0$*,*

(1) $(N)_{m,n} \Rightarrow (L)_{n-m}$*;* $(L)_m \Rightarrow (N)_{m+4,0}$*;* $(L)_{-m} \Rightarrow (N)_{4,m}$*, and*
(2) *if* f *is surjective, then* $(L)^m \Rightarrow (N)^{m+4,0}$*;* $(L)^{-m} \Rightarrow (N)^{4,m}$*;* $(N)^{m,n} \Rightarrow (L)^{m-n}$*.*

We have the Lipschitz invariance:

Theorem 8. *Let* $f : X \to Y$ *be a map between* σ*-compact spaces* X *and* Y *with normal sequences* $\mathbb{U} = \{\mathfrak{U}_i\}$ *and* $\mathbb{V} = \{V_i\}$*, respectively, and let* F *be a subset of* X*. For* $m \ge 0$*, consider the properties:*

$\mathbf{(H)}_m$**:** $\mathrm{H}^s_{\mathbb{V}}(f(F)) \le 3^{ms}\,\mathrm{H}^s_{\mathbb{U}}(F)$ *for* $s \ge 0$*, and*
$\mathbf{(H)}^m$**:** $\mathrm{H}^s_{\mathbb{U}}(F) \le 3^{ms}\,\mathrm{H}^s_{\mathbb{V}}(f(F))$ *for* $s \ge 0$*.*

Then for $m \ge 0$*,*

(1) $(L)_m \Rightarrow (H)_{m+4} \Rightarrow \dim_H^{\mathbb{V}} f(F) \le \dim_H^{\mathbb{U}} F$*.*
(2) $(L)^m \Rightarrow (H)^{m+4} \Rightarrow \dim_H^{\mathbb{V}} f(F) \ge \dim_H^{\mathbb{U}} F$*.*

Corollary 9. *If* $f : X \to Y$ *is a* $(\mathbb{U},\mathbb{V})$*-bi-Lipschitz map. Then for any subset* F *of* X*, Then* $\dim_H^{\mathbb{V}} f(F) = \dim_H^{\mathbb{U}} F$*.*

3. An inverse system approach

If $\boldsymbol{p} = (p_i) : X \to \boldsymbol{X} = (X_i, \mathfrak{U}_i, p_{i,i+1})$ is an approximate resolution of a compact space X, for each $s \geq 0$ and $i \in \mathbb{N}$, we define $\mathrm{H}^s_i(\boldsymbol{p})$ as

$$\inf \left\{ \sum_{k=1}^{n} \left(\frac{1}{3^{i_k}} \right)^s : p_j(X) \subseteq \bigcup_{k=1}^{n} p_{i_k j}(U_{i_k}), U_{i_k} \in \mathfrak{U}_{i_k}, i \leq i_k \leq j, n \in \mathbb{N} \right\}$$

and define the *s-dimensional Hausdorff measure* of $\boldsymbol{p}$ as $\mathrm{H}^s(\boldsymbol{p}) = \lim_{i\to\infty} \mathrm{H}^s_i(\boldsymbol{p})$. Similarly to $\dim^{\mathbb{U}}_H$, we can define the *Hausdorff dimension of* $\boldsymbol{p}$ as $\dim_H(\boldsymbol{p}) = \sup\{s : \mathrm{H}^s(\boldsymbol{p}) = \infty\}$.

Lemma 10. *Let $\boldsymbol{p} = (p_i) : X \to \boldsymbol{X} = (X_i, \mathfrak{U}_i, p_{i,i+1})$ be a normal approximate resolution, and let $\mathbb{U} = \{p_i^{-1}(\mathfrak{U}_i)\}$. Then for each $s \geq 0$, $\mathrm{H}^s(\boldsymbol{p}) = \mathrm{H}^s_{\mathbb{U}}(X)$.*

For each approximate sequence $\boldsymbol{X} = (X_i, \mathfrak{U}_i, p_{i,i+1})$ and for each i, we define $\mathrm{H}^s_i(\boldsymbol{X})$ as

$$\inf \left\{ \sum_{k=1}^{n} \left(\frac{1}{3^{i_k}} \right)^s : X_j \subseteq \bigcup_{k=1}^{n} p^{-1}_{i_k j}(U_{i_k}), U_{i_k} \in \mathfrak{U}_{i_k}, i \leq i_k \leq j, n \in \mathbb{N} \right\}$$

and define the *s-dimensional Hausdorff measure of* $\boldsymbol{X}$ as $\mathrm{H}^s(\boldsymbol{X}) = \lim_{i\to\infty} \mathrm{H}^s_i(\boldsymbol{X})$. Similarly to $\dim_H(\boldsymbol{p})$, we can define the *Hausdorff dimension* $\dim_H \boldsymbol{X}$ *of* $\boldsymbol{X}$ as $\dim_H \boldsymbol{X} = \sup\{s : \mathrm{H}^s(\boldsymbol{X}) = \infty\}$. Note here that the definition of $\mathrm{H}^s_i(\boldsymbol{X})$ does not depend on the projection maps p_i.

Lemma 11. *Let $\boldsymbol{p} = (p_i) : X \to \boldsymbol{X} = (X_i, \mathfrak{U}_i, p_{i,i+1})$ be a normal approximate resolution such that*

$$\operatorname{st} \mathfrak{U}_j < p^{-1}_{ij} \mathfrak{U}_i \text{ for } i < j, \tag{1}$$

and let F be a compact subset of X. For each i, let F_i be a compact polyhedron such that

$$\overline{\operatorname{st}(p_i(F), \mathfrak{U}_i)} \subseteq F_i \subseteq \operatorname{st}(p_i(F), \operatorname{st} \mathfrak{U}_i).$$

Then

1. *$\boldsymbol{F} = (F_i, \mathfrak{U}_i|F_i, p_{i,i+1}|F_{i+1})$ is an approximate sequence, and $\boldsymbol{p}|F = (p_i|F) : F \to \boldsymbol{F} = (F_i, \mathfrak{U}_i|F_i, p_{i,i+1}|F_{i+1})$ is a normal approximate resolution; and*
2. *if $\mathbb{U} = \{p_i^{-1}(\mathfrak{U}_i)\}$, then $\mathrm{H}^s_{\mathbb{U}}(F) = \mathrm{H}^s(\boldsymbol{F})$ for each $s \geq 0$.*

Remark. Given any normal approximate resolution $\boldsymbol{p} : X \to \boldsymbol{X}$ of X, we can always find a normal approximate resolution $\boldsymbol{p}' : X \to \boldsymbol{X}'$ of X so that $\boldsymbol{X}'$ is a subsystem of $\boldsymbol{X}$ and has property (1).

By Lemmas 11 and 10, we have characterizations of $\dim^{\mathbb{U}}_H$ in terms of an approximate sequence and in terms of an approximate resolution.

Theorem 12. *Under the same setting as in Lemma 11, $\dim_H \boldsymbol{F} = \dim^{\mathbb{U}}_H F = \dim_H(\boldsymbol{p}|F)$.*

Remark. All the results in this section hold for the noncommutative versions of approximate sequences and approximate resolutions (limits) in the sense of [4, 5].

It is well-known that if X is a compact metrizable space with covering dimension n, then X can be embedded in I^{2n+1} [3, Theorem V 2]. Motivated by this result, we consider the following question: For each $r > 0$, find the least integer N for which a Cantor set with Hausdorff dimension r can be realized in the cube $[0,1]^N$.

For each $N \in \mathbb{N}$ and for each $i \in \mathbb{N}$, let $\mathrm{I}_i^N = \mathrm{I}^N$ with the usual metric d, let $\mathfrak{U}_i$ be the open covering by open $\frac{1}{3^{i+1}}$-balls, and let $q_{i,i+1} : \mathrm{I}_{i+1}^N \to \mathrm{I}_i^N$ be the identity map. Then it is easy to see that $\boldsymbol{I}^N = (\mathrm{I}_i^N, \mathfrak{U}_i, q_{i,i+1})$ is an approximate sequence. For each $i \geq 1$, let $q_i : \mathrm{I}^N \to \mathrm{I}_i^N$ be the identity map. Then the approximate map $\boldsymbol{q} = (q_i) : \mathrm{I}^N \to \boldsymbol{I}^N$ is a normal approximate resolution of I^N, and the metric $\mathrm{d}_{\boldsymbol{q}}$ induced by $\boldsymbol{q}$ is isometric to the metric d on I^N.

Theorem 13. *For each positive real number r, let*

$$N = \left[\frac{\log 3}{\log 2}(r+1) + 1\right].$$

Then there exist a Cantor set X in I^N and compact subsets X_i of I_i^N so that the restriction $\boldsymbol{p} = (q_i|X) : X \to \boldsymbol{X} = (X_i, \mathfrak{U}_i|X_i, q_{i,i+1}|X_{i+1})$ is a normal approximate resolution of X, and $\dim_H(\boldsymbol{p}) = r$. Here, for each $r > 0$, let $[r]$ denote the least integer that is greater than or equal to r.

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(T. Miyata) Division of Mathematics and Informatics, Faculty of Human Development, Kobe University, 3-11 Tsurukabuto, Nada-Ku, Kobe, 657-8501 JAPAN

E-mail address, T. Miyata: tmiyata@kobe-u.ac.jp

(T. Watanabe) Department of Mathematics and Information Sciences, Faculty of Education, Yamaguchi University, Yamaguchi-City, 753-8513 JAPAN

E-mail address, T. Watanabe: tadashi@po.yb.cc.yamaguchi-u.ac.jp

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ON STRONGLY σ-SHORT BOOLEAN ALGEBRAS

MAKOTO TAKAHASHI

ABSTRACT. We investigate strongly σ-shortness of some Boolean algebras. Especially, we show that every (κ, ω)-caliber Boolean algebra of density $\geq \kappa$ is not strongly σ-short.

1. INTRODUCTION

In [5], we introduced σ-shortness and strongly σ-shortness of Boolean algebras. We say that a subset D of a Boolean algebra $\mathbf{B}$ is *σ-short* if every strictly descending sequence of length ω in D does not have a nonzero lower bound in $\mathbf{B}$, *$\wedge$-closed* if $d_1 \wedge d_2 \in D$ for every $d_1, d_2 \in D$ such that $d_1 \wedge d_2 > \mathbf{0}$. $\mathbf{B}$ is said to be *σ-short* if it has a σ-short dense subset and *strongly σ-short* if it has a σ-short $\wedge$-closed dense subset. We note that $\mathbf{B}$ itself is not a σ-short set if $\mathbf{B}$ is atomless. In this paper, we assume that Boolean algebras are atomless. Typical examples of σ-short Boolean algebras are regularly filtered Boolean algebras which are also strongly σ-short. Another examples of σ-short Boolean algebras are measure algebras. In [5], we left the following problems:

(1) Are measure algebras strongly σ-short?
(2) Is Prikry forcing σ-short?

After the author lectured in the 2002 General Topology Symposium, Jörg Brendle showed the following theorem which concerns to the first problem.

Theorem A(Brendle). *Let B_κ be the algebra for adding κ many random reals.*

(1) *B_ω is not strongly σ-short.*
(2) *Assume that the density of B_κ equals to κ. Then B_κ is strongly σ–short.*

Yasuo Yoshinobu and the author extend the first result more general as follows.

Theorem 1. *Suppose that $\mathbf{B}$ satisfies (κ, ω)-caliber and $d(B) \geq \kappa$ where $d(B)$ denotes the density of B. Then $\mathbf{B}$ is not strongly σ-short.*

Let κ be a measurable cardinal, and U a normal measure on κ. Let $\mathbb{P}_U$ denote the canonical poset of the Prikry forcing associated with U and $\mathbf{B}_U$ be the Boolean completion of $\mathbb{P}_U$. Since $\mathbf{B}_U$ satisfies (κ, ω)–caliber and $d(\mathbf{B}_U) \geq \kappa$, $\mathbf{B}_U$ is not strongly σ-short by virtue of the thorem above. Y. Yoshinobu also show that Prikry forcing itself is not σ-short. However, it is still open whether Prikry algebra $\mathbf{B}_U$ is not σ-short.

I would like to thank Jörg Brendle and Yasuo Yoshinobu for many valuable comments. In this paper we give proofs of results above with their permission.

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2. Preliminaries

In this section we give basic definitions, notation and results which are needed in this paper. The reader is assumed to be familiar with the basic facts about Boolean algebras found in [3]. For basic facts about set theory, we refer to [2] and [4].

We use the letter κ for infinite cardinals; the letters α, β for ordinals; the letters $\mathbf{A}$, $\mathbf{B}$ for infinite atomless Boolean algebras. For a Boolean algebra $\mathbf{B}$, we denote by $\mathbf{B}^+$ the set of all nonzero elements of $\mathbf{B}$. We use $\wedge, \vee, -$ for Boolean operations. $\mathbf{A} \leq \mathbf{B}$ means that $\mathbf{A}$ is a subalgebra of $\mathbf{B}$. If $X \subset \mathbf{A}$, then $\langle X \rangle_{\mathbf{A}}$ is the subalgebra of $\mathbf{A}$ generated by X. We omit the subscript if there is no confusion. We say that a set $D \subseteq \mathbf{B}^+$ is dense if for every $b \in \mathbf{B}^+$ there exists $d \in D$ such that $d \leq b$. For a poset $\mathbf{P}$, we denote by $B(\mathbf{P})$ the Boolean completion of $\mathbf{P}$. For a set X, we denote by $|X|$ the cardinality of X. For a Boolean algebra $\mathbf{B}$, we define the *density* $d(\mathbf{A})$ of $\mathbf{B}$ by $d(\mathbf{A}) = \min\{|X| \mid X is dense in \mathbf{B}\}$. For a set X and a cardinal κ, let $[X]^\kappa = \{Y \subseteq X \mid |Y| = \kappa\}$. $\mathcal{C} \subseteq [X]^\kappa$ is said to be λ-*closed* if it is closed with respect to union of increasing sequences of length $\leq \kappa$. $\mathcal{C} \subseteq [X]^\kappa$ is said to be *unbounded* if it is cofinal in $([X]^\kappa, \subseteq)$. We say that a Boolean algebra B has (κ, ω)-caliber if for any uncountable subset $T \subseteq B$ of size κ, there is countable $F \subseteq T$ such that F has a non-zero lower bound in B. It is well-known that the random algebra has (ω_1, ω)-caliber.

$\mathbf{A}$ is called a *regular* subalgebra of $\mathbf{B}$, in symbol $\mathbf{A} \leq_{reg} \mathbf{B}$, whenever for every $M \subseteq \mathbf{A}$, $\bigwedge^{\mathbf{A}} M = \mathbf{0}$ implies $\bigwedge^{\mathbf{B}} M = \mathbf{0}$. $\mathbf{B}$ is said to be *regularly filtered* if $\{\mathbf{A} \leq_{reg} \mathbf{B} \mid |\mathbf{A}| \leq \aleph_0\}$ contains an $\aleph_0$-closed unbounded subset of $[\mathbf{B}]^{\aleph_0}$. Basic facts about regularly filtered Boolean algebras can be found in [1]

3. Examples

In this section, we shall give some examples of σ-short Boolean algebras.

Example 1. For any set X, let $\mathrm{Fr}\, X$ be the free Boolean algebra over X. We assume without loss of generality that $X \subset \mathrm{Fr}\, X$. Put

$$D = \{\pm x_1 \cdot \pm x_2 \cdot \ldots \cdot \pm x_n \mid n \in \omega, x_1, x_2, \cdots, x_n \in X\} - \{0\}.$$

Clearly, D is a σ-short $\wedge$-closed dense subset of $\mathrm{Fr}\, X$. Hence, $\mathrm{Fr}\, X$ is strongly σ-short.

Example 2. Let $(\mathbf{B}, \mu)$ be a measure algebra. Put

$$D = \{a \in \mathbf{B} \mid \mu(a) = \frac{1}{n+1} \text{ for some } n \in \omega\}.$$

Then D is a σ-short dense subset of $\mathbf{B}$. Hence $(\mathbf{B}, \mu)$ is σ-short. $(\mathbf{B}, \mu)$ is not regularly filtered. In fact, measure algebras are weakly σ-distributive but free Boolean algebras are not weakly σ-distributive, and a regular subalgebra of a weakly σ-distributive Boolean algebra is again weakly σ-distributive. Thus $(\mathbf{B}, \mu)$ does not have any countable atomless regular subalgebra.

Example 3. Let $(\mathbf{P}, \leq)$ be a notion of forcing with finite conditions such that $p \geq q$ if and only if $p \subset q$ for every $p, q \in \mathbf{P}$. $\mathbf{P}$ itself is σ-short and is a σ-short dense subset of $B(\mathbf{P})$. Moreover, if any two compatible elements of $\mathbf{P}$ have an infimum in $\mathbf{P}$, then $\mathbf{P}$ is $\wedge$-closed in $B(\mathbf{P})$, so that $B(\mathbf{P})$ is strongly σ-short.

Every regularly filtered Boolean algebra is strongly σ-short(see [5]). The following example, however, shows that the converse is not true even if it is ccc.

Example 4. Let $S \subset \omega_1$ be a stationary co-stationary set and $(\mathbf{T}, <_{\mathbf{T}})$ be the tree of closed subset of S under end extension. Let $\mathbf{P_T}$ be the set of all finite antichains of $\mathbf{T}$ ordered by reversed inclusion. Then $\mathbf{P_T}$ is a σ-short $\wedge$-closed dense subset of $B(\mathbf{P_T})$ by Example 3, so that $B(\mathbf{P_T})$ is strongly σ-short. $\mathbf{P_T}$ is ccc but not absolutely ccc(see [6]). Since every regularly filtered Boolean algebra is absolutely ccc, $B(\mathbf{P_T})$ is not regularly filtered.

4. STRONGLY σ-SHORT BOOLEAN ALGEBRAS

In this section, we prove main results.

Theorem 1. *Suppose that* $\mathbf{B}$ *satisfies* (κ, ω)*-caliber and* $d(\mathbf{B}) \geq \kappa$*. Then* $\mathbf{B}$ *is not strongly* σ*-short.*

Proof. Suppose that $\mathbf{B}$ is strongly σ-short. Let D be a dense, σ-short and $\wedge$–closed subset of $\mathbf{B}$. There is $D_0 \subseteq D$ dense, $|D_0| = d(\mathbf{B})$, which is still σ-short and $\wedge$-closed. Enumerate $D_0 = \{d_\alpha;\ \alpha < d(\mathbf{B})\}$. We recursively construct sets D_1^α $(\alpha < d(\mathbf{B}))$and X such that

(1) $D_1^\alpha \subseteq D_1^\beta$ for $\alpha < \beta$
(2) $|D_1^\alpha| \leq |\alpha| \cdot \omega < d(\mathbf{B})$
(3) $D_1^\alpha \subseteq D_0$ is $\wedge$–closed
(4) for $\alpha < \beta$, if $d \in D_1^\alpha$ and $e \in D_1^\beta \setminus D_1^\alpha$, then $e \not\geq d$.
(5) $\forall \alpha < d(\mathbf{B}) \exists x \in D_1^{\alpha+1}[x \leq d_\alpha]$
(6) $\forall \alpha < d(\mathbf{B})$, $\alpha \in X$ if and only if there is no $d \in D_1^\alpha$ such that $d_\alpha \geq d$

Let $D_1^0 = \emptyset$ and $X_0 = \emptyset$. For a limit ordinal λ, let $D_1^\lambda = \bigcup_{\alpha<\lambda} D_1^\alpha$ and $X_\lambda = \bigcup_{\alpha<\lambda} X_\alpha$. Let $\alpha = \beta + 1$ be a successor ordinal. If there is $d \in D_1^\beta$ such that $d_\beta \geq d$, let $D_1^\alpha = D_1^\beta$ and $X_\alpha = X_\beta$. If there is no $d \in D_1^\beta$ such that $d_\beta \geq d$, let D_1^α be the $\wedge$–closure of $D_1^\beta \cup \{d_\beta\}$ and $X_\alpha = X_\beta \cup \{\beta\}$. Put $X = \bigcup_{\alpha<d(B)} X_\alpha$ and $D_1 = \bigcup_{\alpha<d(\mathbf{B})} D_1^\alpha$. Since $D_1 \subseteq D_0$, D_1 is σ-short. It is easy to see that D_1 is dense and $\wedge$–closed.

Since $d(\mathbf{B}) \geq \kappa$ and X is cofinal in $d(\mathbf{B})$, $|X| \geq \kappa$. Since $\mathbf{B}$ satisfies (κ, ω)-caliber, there exists countable subset F of X such that $\{d_\alpha | \alpha \in F\}$ has a non-zero lower bound b in $\mathbf{B}$. Without loss of generality, we assume $F = \{\alpha_n | n \in \omega\}$ and $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \cdots$. Since $\alpha_n \in X$, $d_{\alpha_n} \not\geq d_{\alpha_1} \wedge d_{\alpha_2} \wedge ... \wedge d_{\alpha_{n-1}}$. Hence the sequence of $e_n = d_{\alpha_1} \wedge d_{\alpha_2} \wedge ... \wedge d_{\alpha_n} \in D_1$ is strictly decreasing. Since $d_{\alpha_n} \geq b$ for every $n \in \omega$, $\{e_n | n \in \omega\}$ has a non-zero lower bound $\mathbf{B}$. So D_1 is not σ-short. This contradicts that D_0 is σ-short. □

Let $\mathbf{B}_\kappa$ be the algebra for adding κ many random reals. Since $\mathbf{B}_\omega$ satisfies (ω_1, ω)-caliber and $d(\mathbf{B}_\omega) \geq \omega_1$, we have the following:

Corollary 1 (Brendle)**.** $\mathbf{B}_\omega$ *is not strongly* σ*-short.*

On the other hand, J. Brendle also showed that

Theorem 2 (Brendle)**.** *Assume that* $d(\mathbf{B}_\kappa) = \kappa$*. Then* $\mathbf{B}_\kappa$ *is strongly* σ*–short.*

Proof. Let $D \subseteq \mathbf{B}_\kappa$ be dense, $|D| = \kappa$. Say $D = \{b_\alpha;\ \alpha < \kappa\}$. For each α choose $\gamma_\alpha \notin \mathsf{supp}(b_\alpha)$ in such a way that the γ_α are distinct for distinct α. Let $a_\alpha = b_\alpha \wedge [\{\langle\langle\gamma_\alpha, 0\rangle, 0\rangle\}]$. Here $\{\langle\langle\gamma_\alpha, 0\rangle, 0\rangle\}$ denotes the partial function $p : \kappa \times \omega \to 2$ with domain the singleton $\{\langle\gamma_\alpha, 0\rangle\}$ and $p(\langle\gamma_\alpha, 0\rangle) = 0$. $[p]$ is the open set defined by p. Let D' be the $\wedge$–closure of the collection of the a_α. Assume $\{d_n;\ n \in \omega\} \subseteq D'$

is strictly decreasing. Each d_n is a finite conjunction of a_α, say $d_n = \bigwedge_{i<k_n} a_{\alpha_{i,n}}$. If $\{a_{\alpha_{i,n}};\ i < k_n, n \in \omega\}$ is infinite, $\bigwedge_n d_n = 0$ is immediate. If $\{a_{\alpha_{i,n}};\ i < k_n, n \in \omega\}$ is finite, then the sequence cannot be strictly decreasing, a contradiction. So D' is σ–short.

□

Let κ be a measurable cardinal, and U a normal measure on κ. Let $\mathbb{P}_U$ denote the canonical poset of the Prikry forcing associated with U, that is, $\mathbb{P}_U$ consists of all pairs (s, A) satisfying

(1) s is a finite strictly increasing sequence of ordinals below κ,
(2) $A \in U$ and $\max s < \min A$ (consider $\max \emptyset = -1$),

and $(s, A) \leq (t, B)$ iff

(1) s is an end extension of t,
(2) $s \setminus t \subseteq B$, and $A \subseteq B$.

Let $\mathbf{B}$ be the Boolean completion of $\mathbb{P}_U$. Since $\mathbf{B}$ satisfies (κ, ω)–caliber and $d(\mathbf{B}) \geq \kappa$, $\mathbf{B}$ is not strongly σ-short. It is open whether $\mathbf{B}$ is σ-short. However, Y. Yoshinobu showed that $\mathbb{P}_U$ itself is not σ-short.

Claim 1. *Whenever D is a dense subset of $\mathbb{P}_U$, there exists s such that*

$$\forall X \in U \exists Y \in U [Y \subseteq X \wedge (s, Y) \in D].$$

Proof. Suppose not. Then for every s there is $X_s \in U$ such that

$$\forall Y \in U [Y \subseteq X_s \to (s, Y) \notin D].$$

For each $-1 \leq \alpha < \kappa$, set

$$X_\alpha := \bigcap_{\max s = \alpha} X_s$$

and

$$X := \triangle_{-1 \leq \alpha < \kappa} X_\alpha.$$

Note that $X \in U$ and thus $(\emptyset, X) \in \mathbb{P}_U$. Therefore there is $(s, Y) \in D$ such that $(s, Y) \leq (\emptyset, X)$. But then for every $\alpha \in Y$, $\alpha \in X_{\max s} \subseteq X_s$ and thus $Y \subseteq X_s$ holds. Contradiction. □

Theorem 3. *Prikry forcing is not σ-short.*

Proof. Pick s as in the above claim, and pick $X_0 \in U$ such that $(s, X_0) \in D$. Whenever X_n is given, pick $X_{n+1} \in U$ such that $X_{n+1} \subseteq X_n \setminus \{\min X_n\}$ and $(s, X_{n+1}) \in D$. Then (s, X_n)'s form a strictly decreasing ω-sequence in D which has a common extension. This shows that D is not σ-short. □

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E-mail address: makoto@kobe-u.ac.jp

Faculty of Human Development, Kobe University, Tsurukabuto, Nada, Kobe 657-8501, Japan

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SMOOTHNESS OF SCALING FUNCTIONS AND TRANSFER OPERATORS

TATSUHIKO YAGASAKI

この論説は，rank M 直交 スケール関数 と それに付随する ウェーブレット関数の滑らかさ を Transfer operator の spectral radius を用いて評価するという 米谷・中岡・大倉氏 との共同研究 [6] の紹介を目的としたものである．

1. ウェーブレット解析について

1.1. Fourier 解析 から Wavelet 解析 へ.

Fourier 解析 から Wavelet 解析 への移行は，数学的には 次の様に理解される [5]. (複素)関数空間 $L^2([0,2\pi])$ は，直交基底 e^{inx} $(n \in \mathbb{Z})$ を持ち，任意の関数 $f \in L^2([0,2\pi])$ は Fourier 級数展開

$$f(x) = \sum_{n\in\mathbb{Z}} c_n e^{inx}$$

を持つ．Fourier 係数 c_n は

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\,dx.$$

で与えられる．また，Fourier 変換 $\mathcal{F} : L^2([0,2\pi]) \to L^2([0,2\pi])$ が次式で定義される：

$$\mathcal{F}(f)(\xi) = \int_{-\infty}^{\infty} f(x)e^{-i\xi x}\,dx.$$

基底 $\{e^{inx}\}$ の特徴は，1つの基本関数 e^{ix} の dilation によって構成されていることである．これに対し，基本的な関数空間である $L^2(\mathbb{R})$ も このような基本関数 $\psi(x)$ を持つかどうか が自然な問題として生じる．$\mathbb{R}$ は (非コンパクト) 線形空間 であるから，基本関数に対する変換として，dilation に加えて translation $x+k$ も考慮に入れるのは自然なことである．dilation を 整数倍 nx ではなく 2 の整数乗倍 2^jx にとると，問題は 次の様に述べられる：

問題 1.1. 関数 $\psi(x) \in L^2(\mathbb{R})$ で，$\psi_{j,k}(x) = 2^{j/2}\psi(2^jx-k)$ $(j,k \in \mathbb{Z})$ が $L^2(\mathbb{R})$ の正規直交基底となるものが存在するか？

この条件を満たす基本関数 $\psi(x)$ を 2進 直交 ウェーブレット と呼ぶ．$\psi(x)$ はコンパクト台を持つことが望ましい．最も簡単な例は，次で定義される Harr 関数 である：

$$\psi_H(x) = \begin{cases} 1 & (0 \le x < 1/2) \\ -1 & (1/2 \le x < 1) \\ 0 & (\text{その他}). \end{cases}$$

$\psi(x)$ が 2進 直交 ウェーブレット のとき，関数 $f \in L^2(\mathbb{R})$ に対し，展開

$$f = \sum_{j,k\in\mathbb{Z}} c_{j,k}\psi_{j,k}$$

を f の ウェーブレット展開と呼ぶ．ウェーブレット係数 $c_{j,k}$ は，次式で与えられる：

$$c_{j,k} = \int_{-\infty}^{\infty} f(x)\overline{\psi_{j,k}(x)}\,dx = (\tfrac{1}{2^j})^{-\frac{1}{2}} \int_{-\infty}^{\infty} f(x)\overline{\psi\left(\frac{x-\frac{k}{2^j}}{\frac{1}{2^j}}\right)}\,dx.$$

$\left(\dfrac{k}{2^j}, \dfrac{1}{2^j}\right)$ を 連続変数 (b,a) で置き換えて ウェーブレット変換 $W_\psi : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ が次式で定義される：

$$W_\psi(f)(b,a) = |a|^{-\frac{1}{2}} \int_{-\infty}^{\infty} f(x)\overline{\psi\left(\frac{x-b}{a}\right)}\,dx.$$

1.2. **多重解像度解析.**
各 $j \in \mathbb{Z}$ に対して $L^2(\mathbb{R})$ の閉部分空間 W_j を

$$W_j := cl_{L^2(\mathbb{R})}\langle \psi_{j,k} \,:\, k \in \mathbb{Z}\rangle$$

で定めると，次が成り立つ：

$$L^2(\mathbb{R}) = \oplus_{j\in\mathbb{Z}} W_j \qquad \text{(直交直和)}.$$

さらに $V_j := \oplus_{i\le j-1} W_i$ と置くと，$\{V_j\}$ は次の条件を満たす：

(1) $V_j \subset V_{j+1} \quad (j \in \mathbb{Z})$
(2) $cl_{L^2(\mathbb{R})}\left(\cup_{j\in\mathbb{Z}} V_j\right) = L^2(\mathbb{R})$
(3) $\cap_{j\in\mathbb{Z}} V_j = \{0\}$
(4) $f(x) \in V_j \iff f(2x) \in V_{j+1} \quad (j \in \mathbb{Z})$

この条件を満たす閉部分空間の列 $\{V_j\}$ を 一般に 2進 直交 MRA (多重解像度解析) と呼ぶ.

逆に，このような 直交 MRA $\{V_j\}$ を基に，ウェーブレット $\psi(x)$ を構成するという観点に立つと，次の条件を満たす関数 $\varphi(x) \in L^2(\mathbb{R})$ の存在を要請することは自然である：

(∗) $\varphi(x-k)$ $(k \in \mathbb{Z})$ は V_0 の正規直交基底.

この条件を満たす関数 $\varphi(x)$ を MRA $\{V_j\}$ を生成する 2進 直交 スケール関数 と呼ぶ．$\varphi_{j,k}(x) = 2^{j/2}\varphi(2^j x - k)$ $(j,k \in \mathbb{Z})$ と置くと，(4) より $\varphi_{j,k}(x)$ $(k \in \mathbb{Z})$ は V_j の正規直交基底 になる．また，$\varphi_{1,k}$ $(k \in \mathbb{Z})$ は V_1 の直交基底で，$\varphi \in V_0 \subset V_1$ であるから，ある $\{a_k\} \in \ell^2$ が存在して

$$\varphi(x) = \sum_{k\in\mathbb{Z}} a_k \varphi(2x-k)$$

と一意に表される．この関係式を 2 - スケール関係 と呼ぶ.

2進 直交 ウェーブレットの具体的な構成では，スケール関数 と ウェーブレット が組として 構成される．従って，上の整合性を持つ 2進 直交 MRA $\{V_j\}$, スケール関数 $\varphi(x)$, ウェーブレット $\psi(x)$ は，1つの系として捉えられるものである．さらに，2進条件は，一般の $M \ge 2$ に対して M 進条件 に自然に拡張される.

以上の考察に基づいて，一般の $M \ge 2$ に対して M 進 直交 MRA の枠組みの中で，M 進 直交 スケール関数 $\varphi(x)$ 及び M 進 直交 ウェーブレット系 $\psi^1(x), \cdots, \psi^{M-1}(x)$ が，次の様に定義される：

定義 1.1. 次の条件を満たす $L^2(\mathbb{R})$ の閉部分空間の列 $\{V_j\}$ を M 進 直交 MRA と呼ぶ：

(1) $V_j \subset V_{j+1} \quad (j \in \mathbb{Z})$
(2) $cl_{L^2(\mathbb{R})}(\cup_{j\in\mathbb{Z}} V_j) = L^2(\mathbb{R})$
(3) $\cap_{j\in\mathbb{Z}} V_j = \{0\}$
(4) $f(x) \in V_j \Longleftrightarrow f(Mx) \in V_{j+1} \quad (j \in \mathbb{Z})$
(5) ある関数 $\varphi(x) \in V_0$ が存在して $\varphi(x-k)$ $(k \in \mathbb{Z})$ は V_0 の正規直交基底.
(6) 直交直和分解 $V_1 = V_0 \oplus (\oplus_{i=1}^{M-1} W_1^i)$ 及び 関数 $\psi^i(x) \in W_1^i$ $(i = 1, \cdots, M-1)$ が存在して，各 $i = 1, \cdots, M-1$ に対して $\psi^i(x-k)$ $(k \in \mathbb{Z})$ は W_1^i の正規直交基底.

関数 $\varphi(x)$ 及び $\psi^1(x), \cdots, \psi^{M-1}(x)$ を各々，この MRA における (M 進 直交) スケール関数 及び ウェーブレット系 と呼ぶ.

この定義における 直交基底 という条件は Riesz 基底 という条件に一般化することができる．また，ウェーブレット変換 W_ψ では，必要に応じて さらに弱い条件の下で 基本関数 ψ を選ぶことができる.

このように定義された コンパクト台を持つ (実) M 進 直交 スケール関数・ウェーブレット系 の完全な記述（具体的な構成法）が，I. Daubechies [1], P. N. Heller [3], 中岡 明 等によって与えられている.

1.3. ウェーブレットの正則性.

次に問題になるのが，構成された 直交 ウェーブレット の性質 である．その 1 つとして 正則性 (滑らかさ) の評価 が上げられる．ウェーブレット系 $\psi^i(x)$ は $\varphi(Mx-k)$ の一次結合として表されるので，この問題に関しては，スケール関数の正則性を考察すれば十分である．P. N. Heller – R. O. Wells, Jr. [4] 等は，Sobolev 指数 s_2 による Hölder 指数 の評価を用い，2 進・3 進 の場合に，包括的に 直交スケール関数の滑らかさの数値的評価を与えている．我々は，s_1 指数 を用いて彼等の評価の一部を改良した [6]．次節以後は，この話題の解説である．直交 ウェーブレット の正則性は，台の幅 や vanishing moments 条件 を上げただけでは，効果的に改善されず，零点条件といった他の条件を加味する必要がある.

1.4. 時間-周波数解析.

以上は，数学的な観点からの考察であったが，工学への応用では，Fourier 解析・ウェーブレット解析 は 時間-周波数解析 と解釈される．入力信号の時系列は，時間 t を変数とする $\mathbb{R}$ 上の関数 $f(t)$ として表される．時間-周波数解析では，入力信号 $f(t)$ の中に どの周波数成分が どれだけ含まれるか を 各時刻の周りで 適当な時間幅で検出しようとする．高周波成分は 短い時間幅で検出できるが，中・低周波成分の検出には，その波長に応じた長さの時間幅が必要となる．Fourier 級数展開 は 一定時間幅での 正弦波 への分解であり，また，Fourier 変換 は $(-\infty, \infty)$ 上での積分である．従って，各時刻の周りでの周波数解析のためには，$f(t)$ を局所化することが必要となる.

コンパクト台を持つウェーブレット $\psi(t)$ を用いた ウェーブレット解析では，ウェーブレット に対する dilation 変換 は 周波数変換 及び それに応じた時間幅変換，translation 変換は，中心時刻変換 と見なされ，これにより得られる 直交基底 $\psi_{j,k}$ $(j, k \in \mathbb{Z})$ や ウェーブレット変換 W_ψ は 時間-周波数解析 に適したものとなる．すでに，工学の諸分野において，ウェーブレット解析は，時間-周波数解析 の主要な手段となっている.

2. RANK M スケール関数 に関する基本事項

本節では rank M スケール関数に関する基本事項を説明し，本論説で考察する問題を具体的に設定する．以下，スケール関数，ウェーブレット関数 等は，すべて実関数の範囲で考える．また，Fourier 変換は (符号を変えて) 次式で定義する：

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{i\xi x}\, dx.$$

定義 2.1. 関数 $\varphi(x) \in L^2(\mathbb{R})$ が次の条件を満たすとき，rank M スケール関数 であるという：

(∗) $\displaystyle\sum_k a_k = M$ を満たす実数列 $\{a_k\}_{k\in\mathbb{Z}}$ が存在して 次が成り立つ：

$$\varphi(x) = \sum_{k\in\mathbb{Z}} a_k \varphi(Mx - k).$$

この関係式をスケール関係，また，数列 $\{a_k\}_k$ を スケール列 と呼ぶ．スケール関係は，$\varphi(x)$ の $x = \frac{i}{M^{n-1}}$ $(i \in \mathbb{Z})$ での値から $x = \frac{i}{M^n}$ $(i \in \mathbb{Z})$ での値が定まることを意味しており，スケール列 $\{a_k\}_k$ が求まれば，関数 $\varphi(x)$ を数値的に求める（グラフを数値的に描く）ことは容易となる．

$\varphi(x)$ の構成という観点からは，$\varphi(x)$ 自身よりも Fourier 変換 $\hat{\varphi}(\xi)$ を扱った方が良い．この際，スケール列に対応するものとして，Fourier 級数

$$A(\xi) := \frac{1}{M}\sum_k a_k e^{ik\xi}$$

が重要になる．これを $\varphi(x)$ のシンボルと呼ぶ．これを用いれば，スケール関係は，次式と同値になる：

$$\hat{\varphi}(\xi) = A(\xi/M)\hat{\varphi}(\xi/M).$$

$\varphi(x)$ がコンパクト台を持つときには，さらに 次式が成り立つ：

$$\hat{\varphi}(\xi) = \hat{\varphi}(0)\prod_{j=1}^{\infty} A(\xi/M^j).$$

スケール関数に関係する概念として，最大次数 N，長さ L，及び，直交性 がある：

(i) ある三角多項式 $Q(\xi)$ を用いて，シンボル $A(\xi)$ が次の様に表記されるとき，スケール関数 $\varphi(x)$ は 次数 N を持つという：

$$A(\xi) = \left(\frac{1 + e^{i\xi} + \cdots + e^{i(M-1)\xi}}{M}\right)^N Q(\xi).$$

$\varphi(x)$ が 次数 N を持つが，$N+1$ は持たないとき，$\varphi(x)$ は 最大次数 N を持つ という．最大次数 N における関数 $Q(\xi)$ を，$\varphi(x)$ の既約シンボルと呼ぶ．

(ii) $\varphi(x)$ の長さ L を次式で定める：

$$L = k_1 - k_0 + 1 \qquad (k_1 = \max\{k \in \mathbb{Z} \mid a_k \neq 0\},\ \ k_0 = \min\{k \in \mathbb{Z} \mid a_k \neq 0\}).$$

$\varphi(x)$ がコンパクト台を持つことは，長さ有限 $(L < \infty)$ と同値になる．

(iii) $\varphi(x)$ が直交であるとは，関数列 $\{\varphi(x-k)\}_{k\in\mathbb{Z}}$ が $L^2(\mathbb{R})$ で直交系を成すことである．このとき，シンボル $A(\xi)$ は，次の "直交条件" を満たす：

$$\sum_{k=1}^{M-1} |A(\xi + 2k\pi/M)|^2 = 1.$$

逆に，シンボル $A(\xi)$ が 直交条件 を満たすとき，長さ有限のスケール関数 $\varphi(x)$ が直交であるための必要十分条件は，$Q(\xi)$ が Cohen 条件 を満たすことである．但し，$Q(\xi)$ が Cohen 条件 を満たす とは，$\xi = 0$ の近傍を含む $\mathbb{R}$ のコンパクト部分集合 F が存在して 次が成り立つことである：

(i) $\mathbb{R}$ 上の任意の 2π 周期 非負可測関数 $f(\xi)$ に対して次の等式が成り立つ：

$$\int_{[-\pi,\pi]} f(\xi)\,d\xi = \int_F f(\xi)\,d\xi.$$

(ii) $Q(\xi) \neq 0 \quad (\xi \in \cup_{j=1}^{\infty} M^{-j}F)$.

例えば，$Q(\xi) \neq 0$ $(\xi \in [0, \pi/M])$ ならば，$Q(\xi)$ は Cohen 条件 を満たす．

定義 2.2. 最大次数 N, 長さ L の rank M (実) スケール関数で そのシンボル $A(\xi)$ が 直交条件 を満たすもの全体のクラスを 記号 $\varphi_{M,N,L}$ で表す．$L \geq MN$ が成り立つ．特に，$L = MN$ の場合，長さ最小であるといい，このときの $\varphi_{M,N,L}$ を $\varphi_{M,N}$ で表す．

$\varphi_{M,N,L}$ の完全な記述（具体的な構成法）が [3] によって与えられている．

定理 2.1. (1) 任意の $\varphi(x) \in \varphi_{M,N,L}$ に対して，$R(\xi) = |Q(\xi)|^2$ は，非負実関数で，次の形をしている：

(i) $L = MN$ (長さ最小) の場合：　$R(\xi) = R_N(\xi)$

但し $R_N(\xi) \equiv \sum_{n=0}^{N-1} c_n (1-\cos\xi)^n$ 　(c_n は M, N, n にのみ依存して定まっているある実定数).

(ii) $L = MN + L_0 + 1$ $(L_0 \geq 0)$ の場合：　$R(\xi) = R_N(\xi) + \tilde{R}(\xi)$

但し $\tilde{R}(\xi) = (1-\cos\xi)^N \sum_{\substack{1\leq n\leq L_0 \\ n\notin M\mathbb{Z}}} \tilde{c}_n \cos n\xi$ 　$(L_0 \notin M\mathbb{Z}, \quad \tilde{c}_n \in \mathbb{R})$.

(2) 逆に，(1) の形をした任意の非負実関数 $R(\xi)$ が与えられれば，次の手順で 長さ有限の rank M スケール関数 $\varphi(x)$ で シンボルが直交条件を満たすもの が構成される：

(i) Riesz lemma より 3 角多項式 $Q(\xi)$ で $|Q(\xi)|^2 = R(\xi)$ を満たすものが取れる．

(ii) $A(\xi) = \left(\frac{1+e^{i\xi}+\cdots+e^{i(M-1)\xi}}{M}\right)^N Q(\xi)$ により $A(\xi)$ が定義され，

$\hat{\varphi}(\xi) = \hat{\varphi}(0)\prod_{j=1}^{\infty} A(\xi/M^j)$ により $\hat{\varphi}(\xi)$ が定まる．これから，$\varphi(x)$ が定義される．

第 1 節 で述べたように，rank M 直交 ウェーブレット の正則性の評価は，付随する直交スケール関数の正則性の評価に帰着する．以下では，次の問題を考察する：

問題 2.1. (1) $\varphi_{M,N}$ の正則性の評価
(2) 長さ L を最小長 MN から少し伸ばし，既約シンボル $Q(\xi)$ に適当な位置で零点を持たせるとき，$\varphi_{M,N,L}$ の正則性は $\varphi_{M,N}$ の正則性からどれだけ改善されるか？

問題 (2) において，$\varphi_{M,N,L}$ に属するスケール関数で，その既約シンボル $Q(\xi)$ が規定された位置に零点を持つものが存在しない場合もあり，規定された零点が実現されるかどうかは，$\varphi_{M,N,L}$ の構成法 (定理 2.1) に基づいて個別に確かめなければならない．$|Q(\xi)|^2$ は $\cos\xi$ の多項式であるから，多項式 $r(x)$ で $r(\cos\xi) = |Q(\xi)|^2$ を満たすものが一意に定まる．定理 2.1 (1) に基づく 多項式 $r(x)$ の記述から，実現可能な $Q(\xi)$ の零点の位置に関して次のことが分かる：

(i) $N \geq 1$, $\xi_1 \in (\frac{3\pi}{5}, \pi)$ に対して $\varphi \in \varphi_{2,N,2N+4}$ で $r(\cos\xi_1) = r'(\cos\xi_1) = 0$ を満たすものが存在する．
(ii) $M \geq 3$, $N \geq 1$ に対して $\varphi \in \varphi_{M,N,MN+2}$ で $r(\cos\pi) = 0$ を満たすものが存在する．
(iii) $M \geq 3$, $N \geq 1$, $\xi_1 \in [\frac{\pi}{2}, \pi]$ ($M = 3$, $\xi_1 = \frac{2\pi}{3}$ を除く) に対して $\varphi \in \varphi_{M,N,MN+3}$ で $r(\cos\xi_1) = r'(\cos\xi_1) = 0$ を満たすものが存在する．

3. s_p 指数 & Transfer operators

本節では，$\varphi_{M,N,L}$ の正則性 を評価する際に必要となる s_p 指数 及び transfer operator の spectral radius についての基本事項を説明する．

3.1. 滑らかさの指標 – Hölder 指数，s_p 指数.

関数 $f(x)$ の滑らかさの指標として Hölder 指数 $\alpha(f)$ を用いることができる．$n \in \mathbb{Z}_{\geq 0}$ に対して，$\mathcal{C}^n$ は n 回連続微分可能な関数のクラスを表す．さらに，$\alpha = n+\sigma$ ($\sigma \in (0,1)$) に対して，クラス $\mathcal{C}^\alpha$ を次で定義する：

$$\mathcal{C}^\alpha := \left\{ f \in \mathcal{C}^n \;\middle|\; \sup_{x \neq y} \frac{|f^{(n)}(x) - f^{(n)}(y)|}{|x-y|^\sigma} < \infty \right\}.$$

関数 $f(x)$ の Hölder 指数 $\alpha(f)$ は，次式で定義される：

$$\alpha(f) := \sup\{\alpha \geq 0 \mid f \in \mathcal{C}^\alpha\}.$$

Sobolev 埋め込み定理 が示唆するように，Hölder 指数は Sobolev 指数で評価される．関数 $f \in L^2(\mathbb{R})$ に対して，その Sobolev 指数 $s(f)$ は 次式で定義される：

$$s(f) := \sup\left\{ s \in \mathbb{R} \;\middle|\; \int_{-\infty}^{\infty} |(1+|\xi|)^s \hat{f}(\xi)|^2 \, d\xi < \infty \right\}.$$

一般の $p > 0$ に対しても，f の s_p 指数 $s_p(f)$ を次式で定義することができる：

$$s_p(f) := \sup\left\{ s \in \mathbb{R} \;\middle|\; \int_{-\infty}^{\infty} |(1+|\xi|)^s \hat{f}(\xi)|^p \, d\xi < \infty \right\}.$$

命題 3.1. $f \in L^2(\mathbb{R})$ がコンパクト台を持つ時，次の不等式が成立する ([6, Proposition 4.1])：

$$s_p(f) - \tfrac{p-1}{p} \leq s_1(f) \leq \alpha(f) \leq s_r(f) \qquad (p \geq 1,\ r \geq 2)$$

記号を簡略化するために，$\varphi_{M,N,L}$ に属する任意のスケール関数に関して成立する場合には，$s_p(\varphi_{M,N,L})$, $\varphi_{M,N,L} \in \mathcal{C}^n$ 等の記号を用いる．

$\varphi_{M,N,L}$ の s_p 指数は，既約シンボルに付随する transfer operator の spectral radius を用いて評価することが出来る．次節では，transfer operator に関する基本事項を説明する．

3.2. Transfer operators & Spectral radius.

Transfer operator は力学系に付随した概念である．ここでは，1次元区間上の折れ線写像 に付随する transfer operator を扱う．

折れ線写像 $\kappa : [0,\pi] \to [0,\pi]$ を次式で定める：

$$\kappa(x) = \begin{cases} M\left(x - \dfrac{2i}{M}\pi\right) & \dfrac{2i}{M}\pi \le x \le \dfrac{2i+1}{M}\pi \\ M\left(\dfrac{2i+2}{M}\pi - x\right) & \dfrac{2i+1}{M}\pi \le x \le \dfrac{2i+2}{M}\pi. \end{cases}$$

一次変換 $\kappa : \left[\dfrac{i}{M}\pi, \dfrac{i+1}{M}\pi\right] \to [0,\pi]$ の逆一次変換を $\theta_i : [0,\pi] \to \left[\dfrac{i}{M}\pi, \dfrac{i+1}{M}\pi\right]$ で表す．

可測関数 q, f に対して，関数 $T_q(f)$, $U_q(f)$ を次式で定義する：

$$T_q(f)(x) := \sum_{i=1}^{M-1} q(\theta_i(x))f(\theta_i(x)), \qquad U_q(f)(x) := q(x)f(\kappa(x)).$$

さらに，${T_q}^j, {U_q}^j$ $(j = 1, 2, \cdots)$ は，それぞれ T_q, U_q の j 回合成作用素 を表す．${T_q}^j$, ${U_q}^j$ は随伴作用素の関係にある．すなわち q, f, g を $[0,\pi]$ 上の有界可測関数とするとき，次式が成り立つ：

$$\int_0^\pi {T_q}^j(f)\, g\, dx = M^j \int_0^\pi f\, {U_q}^j(g)\, dx \qquad (j = 1, 2, \cdots).$$

$L^\infty([0,\pi])$ は，$[0,\pi]$ 上の実数値有界可測関数全体の成す線形空間にノルム

$$\|f\| := \operatorname{ess\,sup}\{|f(x)| \,:\, x \in [0,\pi]\}$$

を与えた Banach 空間を表す．$L^\infty([0,\pi])$ は 次の 2 つの cone を含んでいる：

$$K := \{f \in L^\infty([0,\pi]) \,:\, f \ge 0 \text{ a.e.}\}, \qquad K_0 := \{f \in L^\infty([0,\pi]) \,:\, \operatorname{ess\,inf}_{[0,\pi]} f > 0\}.$$

$q \in K$ に対して，作用素 $T_q : L^\infty([0,\pi]) \to L^\infty([0,\pi])$ を q に付随する transfer operator と呼ぶ．T_q は 有界線形作用素で positive $(T_q(K) \subset K)$ である．T_q の spectral radius $\rho(T_q)$ は 次で定義される：

$$\rho(T_q) := \lim_{n\to\infty} \|{T_q}^n\|^{\frac{1}{n}}.$$

$\rho(T_q) = \lim_{n\to\infty} \|{T_q}^n(f)\|^{\frac{1}{n}}$ $(f \in K_0)$ が成り立ち，$r, q \in K$, $r \le q$ (a.e.) ならば $\rho(T_r) \le \rho(T_q) \le M\|q\|$ となる．

$\mathcal{C}([0,\pi])$ で 連続関数のなす $L^\infty([0,\pi])$ の部分空間を表す．$\mathcal{C}([0,\pi])$ は，次の cone を含んでいる

$$\tilde{K} := K \cap \mathcal{C}([0,\pi]), \quad \tilde{K}_0 := K_0 \cap \mathcal{C}([0,\pi]).$$

$q \in \tilde{K}$ のとき，$T_q(\mathcal{C}([0,\pi])) \subset \mathcal{C}([0,\pi])$ となる．$\rho(T_q|_{\mathcal{C}([0,\pi])}) = \rho(T_q)$ なので，簡単のため $T_q|_{\mathcal{C}([0,\pi])}$ も T_q で表すことにする．もし $q \in \tilde{K}_0$ ならば，任意の $f \in \tilde{K}$, $f \ne 0$ に対して ある $k \ge 1$ が存在して ${T_q}^k(f) \in \tilde{K}_0$ となる．

T_q は，スペクトルについて 次の様な性質を持つ (cf. [6, Section 3])：

定理 3.1. $q \in \tilde{K}_0$ は Hölder 連続とし，$f \in \tilde{K}_0$ とする．$f_n := {T_q}^n(f)$, $g_n := \dfrac{f_n}{\|f_n\|}$ $(n \geq 1)$ と置く．

このとき，次が成り立つ：

(i) 次を満たす $g \in \tilde{K}$ が一意に存在する：　$\|g\| = 1$,　$\lambda > 0$ が存在して $T_q(g) = \lambda g$.
このとき，さらに，$g \in \tilde{K}_0$, $\lambda = \rho(T_q)$ が成り立つ．

(ii) g_n は g に一様収束する．

(iii) $\dfrac{f_{n+1}}{f_n}$ は $\rho(T_q)$ に一様収束する．

(iv) $\min_{[0,\pi]} \dfrac{f_{n+1}}{f_n} \nearrow \rho(T_q)$,　$\max_{[0,\pi]} \dfrac{f_{n+1}}{f_n} \searrow \rho(T_q)$.

命題 3.2. α, β を正定数とする．$p, q \in K_0$ で $\alpha \leq p, q \leq \beta$ ならば 次式 が成立する：

$$|\rho(T_p) - \rho(T_q)| \leq M\frac{\alpha}{\beta}\|p - q\|.$$

4. $\varphi_{M,N,L}$ の正則性の評価

以下，$\varphi(x) \in \varphi_{M,N,L}$ とし，$Q(\xi)$ を $\varphi(x)$ の既約シンボルとする．

4.1. s_2 指数の評価. — [2, 4]

T. Eirola [2], P. N. Heller – R. O. Wells, Jr. [4] 等は，$q(\xi) = |Q(\xi)|^2$ を weight 関数とする transfer operator $T_{|Q|^2}$ を用いて，$s_2(\varphi)$ の 次の様な表示式を与えた：

定理 4.1.
$$s_2(\varphi) = N - \frac{1}{2}\log_M \rho(T_{|Q|^2}).$$

$|Q|^2$ は $\cos\xi$ の多項式になり，作用素 $T_{|Q|^2}$ は $\cos k\xi$ で張られる 標準的な 有限次元不変部分空間 を持つ．$\rho(T_{|Q|^2})$ は $T_{|Q|^2}$ の この部分空間への制限を表す positive matrix の 最大固有値 に一致する．このことは，$T_{|Q|^2}$ が正値固有関数を持つ事を意味し，上の表示式の証明で重要であると同時に，$s_2(\varphi)$ が行列の固有値の数値計算で容易に求められることも意味する．

$$s_2(\varphi) - \frac{1}{2} \leq \alpha(\varphi) \leq s_2(\varphi)$$

であるから，これにより，φ の正則性が 数値的に評価されることになる．彼等は，$M = 2, 3$ の場合に，この数値計算を行い，

$$\varphi_{2,7} \in \mathcal{C}^2, \quad \varphi_{2,11} \in \mathcal{C}^3, \quad \varphi_{3,9} \in \mathcal{C}^1$$

といった結果や，$s_2(\varphi_{M,N})$ の 上・下からの評価，さらに，$N \to \infty$ のときの漸近的な振る舞い 等 について 包括的に調べている．

長さ最小の場合 $|Q| > 0$ であるが，長さ L を 最小長 MN から少し伸ばせば，$|Q|$ が適当な位置に零点を持ち，$\max |Q|$ が小さくなるような スケール関数 $\varphi(x) \in \varphi_{M,N,L}$ を構成することが出来る．彼等は，$Q(\xi)$ が 写像 $\xi \mapsto M\xi \pmod{2\pi}$ の周期点や準周期点に零点を持つと，最小長の場合に比べて正則性が著しく改善することを示している．

4.2. s_p **指数の評価.** — [6, Section 4]

一般の $p>0$ の場合，weight 関数として $q(\xi):=|Q(\xi)|^p$ をとると，$q(\xi)$ は 2π 周期を持つ Hölder 連続な偶関数となり，特に，$\varphi(x)$ が長さ最小の場合には，$q(\xi)$ は正値 C^∞ 関数で $[0,\pi]$ 上で 狭義 単調増加である．現時点では，$Q(\xi)$ に関する適当な仮定無しで，常に T_q が正固有関数を持つかどうかは不明である．このため，以下の 定理 4.2 の評価式は不等式の形になっている．

任意の $f\in K_0$ に対して $T_q(f)\in K_0$ となり，$\mu(f),\lambda(f)>0$ を

$$\mu(f):=\operatorname{ess\,inf}_{[0,\pi]}\frac{T_q(f)}{f},\qquad \lambda(f):=\operatorname{ess\,sup}_{[0,\pi]}\frac{T_q(f)}{f}$$

で定義することができる．$\mu(f)\le\rho(T_q)\le\lambda(f)$ が成り立つ．

定理 4.2. 次の不等式が成り立つ：

(i) $s_p(\varphi)\ge N-\dfrac{1}{p}\log_M\rho(T_q)$.

(ii) Q が Cohen 条件を満たすとき，$s_p(\varphi)\le N-\dfrac{1}{p}\sup\{\log_M\mu(f)\mid f\in K_0\}\le N$.

系 4.1. $Q(\xi)$ が $[0,\pi]$ に零点を持たないとき，$s_p(\varphi)=N-\dfrac{1}{p}\log_M\rho(T_q)$.

4.3. s_p **指数の数値計算 — 階段関数による近似.** — [6, Section 5]

$s_2(\varphi)-\dfrac{1}{2}\le s_1(\varphi)\le\alpha(\varphi)$ であるから，第 4.1 節で $s_2(\varphi)$ から得られている φ の正則性の評価を，$s_1(\varphi)$ の数値計算により さらに改良できる可能性がある．$\rho(T_q)$ を数値計算で数値的に評価するためには，離散化による有限近似が必要となる．$p=2$ の場合は，T_q が，標準的な有限次元不変部分空間を持ち，$\rho(T_q)$ の数値計算は，この部分空間に制限された T_q を表す行列の固有値の計算に帰着した．しかし，一般の p の場合には，このような 標準的な 有限次元不変部分空間 は見当たらない．そこで，自然に考えられる方法は，q を階段関数で近似する方法である．すなわち，q を階段関数 $q^\pm$ を用いて $q^-\le q\le q^+$ という形に 挟んで $\rho(T_q)$ を $\rho(T_{q^\pm})$ で評価するわけである．

$N\ge 1$ に対して，$S_N([0,\pi])$ で，$[0,\pi]$ を N 等分し，その各小区間上定数となる階段関数全体の成す $L^\infty([0,\pi])$ の N 次元部分空間 を表す．$m\ge 1$ とする．$q\in S_{M^2m}([0,\pi])$, $q\ge 0$ のとき，$T_q(S_{Mm}([0,\pi]))\subset S_{Mm}([0,\pi])$ となる．$q^-,q^+\in S_{M^2m}([0,\pi])$ を，各々 $q^-\le q\le q^+$ a.e. を満たす 最大・最小階段関数とする．$f\in S_{Mm}([0,\pi])$ とする．$f_n^\pm:=T_{q^\pm}^{\,n}(f)\in S_{Mm}([0,\pi])$ $(n=1,2,\cdots)$ と置く．m が十分大きければ $f_n^\pm\in K_0$ となり $\mu_n^\pm,\lambda_n^\pm$ を次の様に定義することができる：

$$\mu_n^\pm:=\operatorname{ess\,inf}_{[0,\pi]}\frac{f_{n+1}^\pm}{f_n^\pm},\qquad \lambda_n^\pm:=\operatorname{ess\,sup}_{[0,\pi]}\frac{f_{n+1}^\pm}{f_n^\pm}\qquad (n=1,2,\cdots).$$

次の関係が成り立つ：$\mu_n^\pm\le\mu_{n+1}^\pm\le\rho(T_{q^\pm})\le\lambda_{n+1}^\pm\le\lambda_n^\pm$.

さらに $q^\pm\in K_0$ のときには，次が成り立つ：$\mu_n^\pm\nearrow\rho(T_{q^\pm})$, $\lambda_n^\pm\searrow\rho(T_{q^\pm})$ $(n\to\infty)$.

Theorem 4.1 より，次が成り立つ：

命題 4.1. (1) m が十分大きいとき，次の不等式が成り立つ：

(i) $s_p(\varphi)\ge N-\dfrac{1}{p}\inf_n\log_M\lambda_n^+$.

(ii) Q が Cohen 条件を満たすとき，$s_p(\varphi)\le N-\dfrac{1}{p}\sup_n\log_M\mu_n^-$.

(2) $Q(\xi)$ が $[0,\pi]$ に零点を持たないとき，次が成り立つ： $\rho(T_q) = \sup_{m,n} \mu_n^- = \inf_{m,n} \lambda_n^+$.

4.4. s_1 **指数の数値計算の結果.** — [6, Section 6]

第 4.3 節の階段関数近似の議論に基づいて s_1 指数の数値計算を行った．その結果と 第 4.1 節の結果を組み合わせることで導かれる $\varphi_{M,N,L}$ の正則性に関する評価の中で，特に，顕著な結果を以下にリストする．($r(x)$ は $|Q(\xi)|^2 = r(\cos\xi)$ を満たす多項式であった.)

(i) $\varphi_{2,62} \in \mathcal{C}^2$, $\varphi_{2,8} \notin \mathcal{C}^2$, $\varphi_{2,9} \in \mathcal{C}^3$, $\varphi_{2,12} \notin \mathcal{C}^4$, $\varphi_{2,13} \in \mathcal{C}^4$;

(ii) $\varphi_{3,4} \in \mathcal{C}^1$, $\varphi_{3,77} \notin \mathcal{C}^2$, $\varphi_{3,78} \in \mathcal{C}^2$;

(iii) $\varphi_{2,6,16} \in \mathcal{C}^3$ if $r(\cos\xi) = r'(\cos\xi) = 0$ for $\xi = \frac{4\pi}{5}$ or $\frac{5\pi}{6}$;
$\varphi_{2,9,22} \in \mathcal{C}^4$ if $r(\cos\xi) = r'(\cos\xi) = 0$ for $\xi = \frac{3\pi}{4}$;
$\varphi_{2,13,30} \in \mathcal{C}^5$ if $r(\cos\xi) = r'(\cos\xi) = 0$ for $\xi = \frac{2\pi}{3}$ or $\frac{3\pi}{4}$;

(iv) $\varphi_{3,3,11} \in \mathcal{C}^1$, $\varphi_{3,6,20} \in \mathcal{C}^2$, $\varphi_{3,9,29} \in \mathcal{C}^3$, $\varphi_{3,13,41} \in \mathcal{C}^4$ if $r(\cos\pi) = 0$;

(v) $\varphi_{3,2,9} \in \mathcal{C}^1$ if $r(\cos\xi) = r'(\cos\xi) = 0$ for $\xi = \frac{5\pi}{6}$;
$\varphi_{3,5,18} \in \mathcal{C}^2$ if $r(\cos\xi) = r'(\cos\xi) = 0$ for $\xi = \frac{5\pi}{6}$ or π;
$\varphi_{3,8,27} \in \mathcal{C}^3$ if $r(\cos\xi) = r'(\cos\xi) = 0$ for $\xi = \pi$;
$\varphi_{3,11,36} \in \mathcal{C}^4$ if $r(\cos\xi) = r'(\cos\xi) = 0$ for $\xi = \pi$;

最後に，$s_2(\varphi_{M,N}) - 1/2$ と $s_1(\varphi_{M,N})$ を比較する数値計算の結果をリストする：

$M = 2$

N	$s_2(\varphi_{2,N}) - 1/2$	$s_1(\varphi_{2,N})$
1	0	0
2	0.5000	0.521
3	0.9150	0.980
4	1.2756	1.392
5	1.5968	1.768
6	1.8884	2.117
7	2.1587	2.442
8	2.4147	2.747
9	2.6617	3.036
10	2.9027	3.310
11	3.1398	3.572
12	3.3740	3.826
13	3.6060	4.072

$M = 3$

N	$s_2(\varphi_{3,N}) - 1/2$	$s_1(\varphi_{3,N})$
1	0	0
2	0.4087	0.443
3	0.6599	0.779
4	0.7950	1.031
5	0.8665	1.211
6	0.9133	1.331
7	0.9499	1.410
8	0.9809	1.462
9	1.0081	1.499
10	1.0323	1.528
11	1.0542	1.552
12	1.0741	1.573
13	1.0925	1.592
77	1.4999	1.999
78	1.5016	2.002

5. 最後に

p が一般の場合，transfer operator T_q の性質は，まだ，十分に理解されたとはいえない．今後，力学系との関連の中で，T_q の spectral radius の性質を明確にしたいと考えている．

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Department of Mathematics, Kyoto Institute of Technology, Matsugasaki, Sakyoku, Kyoto 606, Japan
E-mail address: yagasaki@ipc.kit.ac.jp

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CLOSED IMAGES OF SPACES HAVING g-FUNCTIONS

IWAO YOSHIOKA

1. Introduction and Definitions

In §2, we consider closed images of ks-spaces [34] (ks-spaces are equivalent to k-semistratifiable spaces in the realm of T_2-spaces). Lutzer [22] showed that the closed image of a paracompact k-semistratifiable space (indeed, the image of a k-semistratifiable space by a closed, compact-covering map) is k-semistratifiable. We prove that the closed image of a ks, *Frĕchet* T_1-space or ks, q, regular space is ks and the finite-to-one closed image of a $Nagata$ space is $Nagata$. We also prove that every M_3, q-space is $Nagata$.

In §3, we define the class of weak contraconvergent (=*wcc*) spaces which contain the class of MCP spaces [10] or contraconvergent spaces [31] and are contained in the class of β-spaces. And we prove that the closed images or the pre-images by quasi-perfect maps of *wcc*-spaces are *wcc*-spaces. Also, we prove that the class of *wcc*- and q-spaces are equivalent to the class of wN-spaces. Moreover, we prove that every fiber of a closed map from a *wcc*-space onto a q-space is countably compact.

In §4, we introduce the concept of strongly α-spaces which contain the class of paracompact spaces with G_δ-diagonals and are contained in the class of α-spaces. And we prove that every strongly α, *wcc*-space is k-semistratifiable and every strongly α, *wcc*, $w\theta$-space is metrizable.

In §5, it is showed that quasi-perfect images of γ- ($w\gamma$-) spaces are γ ($w\gamma$) and open closed images of γ- ($w\gamma$-) spaces are γ ($w\gamma$).

Throughout this paper, all maps are onto and we assume no separation axioms unless otherwise stated. The set of natural numbers is denoted by $\mathbf{N}$. Finally, we refer the reader to [6] for undefined terms.

Definition 1.1. For a space X, a structure $(\{g_n(x)\} \mid x \in X)$ is called a g-structure if $g_n(x)$ is an open neighbourhood of x and $g_{n+1}(x) \subset g_n(x)$ for any $x \in X$ and every $n \in \mathbf{N}$. For a subset A of X, we put $g_n(A) = \cup\{g_n(x) \mid x \in A\}$. We now consider the following conditions on a g-structure $\mathcal{G} = (\{g_n(x)\} \mid x \in X)$ of a space X.

(A) If $g_n(x) \cap g_n(x_n) \neq \emptyset (n \geq 1)$, then x is a cluster point of $\{x_n\}_n$.

(B) If $g_n(x) \cap g_n(x_n) \neq \emptyset (n \geq 1)$, then $\{x_n\}_n$ has a cluster point.

(C) If $x \in g_n(x_n)(n \geq 1)$, then $\{x_n\}_n \longrightarrow x$ and if H is closed in X, then $\cap_{n\geq 1}\overline{g_n(H)} = H$.

(D) If $y_n \in g_n(x_n)(n \geq 1)$ and y is a cluster point of $\{y_n\}_n$, then y is a cluster point of $\{x_n\}_n$.

(E) If $y_n \in g_n(x_n)(n \geq 1)$ and $\{y_n\}_n \longrightarrow y$, then $\{x_n\}_n \longrightarrow y$.

(F) If $x \in g_n(x_n)(n \geq 1)$, then $\{x_n\}_n \longrightarrow x$.

(G) If $x \in g_n(x_n)(n \geq 1)$, then $\{x_n\}_n$ has a cluster point.
(H) $\{x\} = \cap_{n\geq 1} g_n(x)$ and, $g_n(y) \subset g_n(x)$ if $y \in g_n(x)$.
(I) If $y_n \in g_n(p)$, $x_n \in g_n(y_n)(n \geq 1)$, then p is a cluster point of $\{x_n\}_n$.
(J) If $y_n \in g_n(p)$, $x_n \in g_n(y_n)(n \geq 1)$, then $\{x_n\}_n$ has a cluster point.
(K) If $y_n \in g_n(p)$, $x_n, p \in g_n(y_n)(n \geq 1)$, then p is a cluster point of $\{x_n\}_n$.
(L) If $y_n \in g_n(p)$, $x_n, p \in g_n(y_n)(n \geq 1)$, then $\{x_n\}_n$ has a cluster point.
(M) If $x_n \in g_n(x)(n \geq 1)$, then $\{x_n\}_n$ has a cluster point.

A T_1-space satisfying (A) $((B))$ is called a *Nagata* space [2, 14] (a wN-space [17]) and $\mathcal{G}$ is called a *Nagata* structure (a wN-structure, respectively). A T_1-space satisfying (C) is called a *stratifiable* space [1, 12] and $\mathcal{G}$ is called a stratifiable structure.

It is well known that [11] a space is stratifiable if and only if it is M_2 [2] and [2] the closed image of a stratifiable space is stratifiable.

A space satisfying (D) $((E))$ is called a *contraconvergent* space [31] (a ks-space [34], which was called a *strongly-quasi-Nagata* space in [18]) and $\mathcal{G}$ is called a contraconvergent structure (a ks-structure, respectively). A space satisfying (F) is called a *semistratifiable* space [5] and $\mathcal{G}$ is called a semistratifiable structure.

Hodel [16] called a β-space (an α-space) for a space satisfying (G) $((H)$, respectively) and proved that a T_2-space is semistratifiable if and only if it is a α- and β-space. It is known that every closed image or finite-to-one open image of a semistratifiable T_2-space is semistratifiable [9: Theorem 2.1].

Moreover, a space satisfying (I) $((J))$ is called a γ-space [17](a $w\gamma$-space [17]) and $\mathcal{G}$ is called a γ-structure (a $w\gamma$-structure, respectively).

A space satisfying (K) $((L))$ is called a θ-space [17] (a $w\theta$-space [17]) and $\mathcal{G}$ is called a θ-structure (a $w\theta$-structure, respectively). Finally, a space satisfying (M) is called a q-space [25] and $\mathcal{G}$ is called a q-structure.

Every $w\theta$-space or wN-space is a q-space.

We define a k-semistratifiable space [22] by a equivalent condition [8: Theorem 3] which is true for no separation axiom.

Definition 1.2. A space X is a *k-semistratifiable* if X has a g-structure $(\{g_n(x)\} \,|\, x \in X)$ such that if $K \cap F = \emptyset$, where K is compact and F is closed, then $K \cap g_m(F) = \emptyset$ for some $m \in \mathbf{N}$.

Proposition 1.3 [34: Proposition 3]. *The following implications hold.*
Nagata spaces $\Longrightarrow$ *stratifiable spaces* $\Longrightarrow$ *contraconvergent spaces*
$\Longrightarrow$ *k-semistratifiable spaces* $\Longrightarrow$ *ks-spaces* $\Longrightarrow$ *semistratifiable spaces.*

Every ks-space is a σ-space [15] and the closed image of a regular σ-space is σ [12: Corollary 4.12], where a space with a σ-locally finite network is called a σ-space [30].

The next result follows from [3: Corollary 3.A.1].

Proposition 1.4. *Every countably compact ks (or γ), T_2-space is compact metrizable.*

Gao [8: Theorem 5] showed that every ks, T_2-space is k-semistratifiable. Here, we give a simpler proof.

Proposition 1.5. *Every ks, T_2-space is k-semistratifiable.*

Proposition 1.6. *Every quasi-perfect map defined on a ks (or γ), T_2- space is perfect.*

2. Contraconvergent spaces

With a view to studying closed images of k-semistratifiable spaces, we first consider the closed images of contraconvergent spaces.

Let f be a map from a space X to a space Y and A be a subset of X. Then by A^*, we denote the subset $\cup\{V : \text{open in } Y \mid f^{-1}(V) \subset A\}$ of Y.

Theorem 2.1. *Let $f : X \longrightarrow Y$ be a closed map. If X is a contraconvergent T_1-space, then Y is contraconvergent.*

Under no separation axiom, we have the following result.

Theorem 2.2. *Let $f : X \longrightarrow Y$ be a quasi-perfect map. If X is a contraconvergent space, then Y is contraconvergent.*

The following example shows that all spaces given in Proposition 1.3, γ-spaces and θ-spaces are not necessarily inverse invariant by perfect maps.

Example 2.3. Let $X = \mathbf{N} \times \beta\mathbf{N}$ ($\mathbf{N}$ has the discrete topology) and $p : X \longrightarrow \mathbf{N}$ be a projection. Then, p is a perfect map and $\mathbf{N}$ is completely metrizable. But, it is easily seen that X is a $wN, w\gamma$-space which is not first countable (thus X is not θ) and not semistratifiable (thus X is not α).

Mizokami and Shimane [28] showed that every k, M_3-space is M_1 . But, the space Y in [26: Example 10.1] is an M_1, *Frĕchet* space which is not $Nagata$. The following theorem shows that every M_3, q-space is $Nagata$. The conditions for wN-spaces to be $Nagata$ are studied in [19].

Theorem 2.4. *For a T_1-space X, the following conditions are equivalent.*
(1) *X is a Nagata space.*
(2) *X is an M_1, q-space.*
(3) *X is a contraconvergent q-space.*
(4) *X is a ks, q, regular space.*

Theorem 2.5. *Every ks, Frĕchet-space is contraconvergent.*

Lutzer [22: Example 4.3] describes that the perfect image of a $Nagata$ space is not even a q-space. On the other hand, every wN-space is preserved by a finite-to-one closed map [10: Proposition 18]. Therefore by Theorems 2.1, 2.4, we have the following result.

Theorem 2.6. *The finite-to-one closed image of a Nagata space is Nagata.*

Remark 2.7. That the finite-to-one closed image of a q-space is q can be directly proved by analogy to the proof of [10: Proposition 18].

By Proposition 1.6 and [22: Proposition 2.5], the quasi-perfect image of a k-semistratifiable T_2-space is k-semistratifiable.

I do not know whether the closed (even perfect) image of a ks, T_1-spaces is ks or not. In [29: Theorem 3.3], Mohamad asserts that the closed image of a regular ks-space is ks. But to me, his proof is not clear.

For closed images of ks-spaces, we have the following results by Theorems 2.1, 2.2, 2.4 and 2.5.

Theorem 2.8. (1) *Let $f : X \longrightarrow Y$ be a closed map. If X is a ks, Frěchet T_1-space or ks, q, regular space, then Y is contraconvergent.*

(2) *Let $f : X \longrightarrow Y$ be a quasi-perfect map. If X is a ks, Frěchet space, then Y is contraconvergent.*

3. Weak contraconvergent spaces

For convenience, we introduce the following notation: if $(A_n)_{n\geq 1}$ and $(B_n)_{n\geq 1}$ are two sequences of subsets, we write $(A_n) \preceq (B_n)$ if $A_n \subset B_n$ for each $n \in \mathbf{N}$.

Definition 3.1 [10]. A space X is said to be *monotonically countably paracopmact* (=MCP) if there exists an operator U assigning to each decreasing sequence $(D_j)_{j\geq 1}$ of closed subsets with empty intersection, a sequence of open subsets $U((D_j)) = (U(n,(D_j)))_{n\geq 1}$ such that

(1) $D_n \subset U(n,(D_j))$ for each $n \in \mathbf{N}$,

(2) $\cap_{n\geq 1}\overline{U(n,(D_j))} = \emptyset$,

(3) if $(D_n) \preceq (E_n)$, then $U((D_j)) \preceq U((E_j))$.

Clearly, every MCP space is countably paracompact.

Definition 3.2. A space X is said to be *weak contraconvergent*(=*wcc*) if there exists a g-structure of X such that if $y_n \in g_n(x_n)(n \geq 1)$ and $\{y_n\}_n$ has a cluster point, then $\{x_n\}_n$ has a cluster point.

It is clear that every contraconvergent space is *wcc*.

Theorem 3.3. *The following implications hold for a T_1-space X.*

(1) *a wN-space*$\Longrightarrow$(2) *an MCP space* $\Longrightarrow$ (3) *a wcc-space* $\Longrightarrow$ (4) *a β-space.*

Example 3.4. (1) There exists a Moore (hence, semistratifiable [17]) space which is neither *wcc* nor ks.

(2) There exists a *wcc*-space which is not semistratifiable.

(3) There exists a *wcc*-space which is not countably paracompact, hence not MCP.

In [10], Good, Knight and Stares showed the results that a space X is wN if and only if it is MCP, q and, X is metrizable if and only if it is MCP, Moore or MCP, γ.

Theorem 3.5. *X is a wN-space if, and only if, it is a wcc, q-space.*

As to metrizations of *wcc*-spaces, we have the following results which can weaken *wcc*-spaces to $quasi$-$Nagata$ spaces [20; 24; 34].

Corollary 3.6. *A T_2-space X is metrizable if it satisfies any one of the following conditions.* (1) *X is a wcc, developable space.*
(2) *X is a wcc, γ-space.*

Remark 3.7. $[0, \omega_1)$ with the order topology is a *wcc*, θ-space which is not metrizable [17: Example 4.12].

In [10: Example 15], it is described that there exists an MCP space which is not preserved by a closed map. On the other hand, weak contraconvergentness is preserved by a closed map. The following two theorems are proved by analogy to the proofs of Theorems 2.1 and 2.2.

Theorem 3.8. *Let $f : X \longrightarrow Y$ be a closed map. If X is a wcc, T_1-space, then Y is wcc.*

Thorem 3.9. *Let $f : X \longrightarrow Y$ be a quasi-perfect map. If X is a wcc-space, then Y is wcc.*

The following example asserts that ks-spaces or *wcc*-spaces are not necessarily preserved by finite-to-one open, compact-covering maps.

Example 3.10. Michael [27: Example 9.1] gave the finite-to-one open, compact-covering map from a completely metrizable space X to a metacompact, locally completely metrizable, non-metrizable Tychonoff space Y which is not Čech-complete. Then Y is a semistratifiable γ, Moore space [9: Theorems 2.1, 4.1; 17: Corollary 4.6]. But if Y is ks or *wcc*, then Y is metrizable by [34: Theorem 3] or Corollary 3.6. This is a contradiction.

Note that every locally metrizable, *wcc*- (or ks-) space is metrizable by [9], [34] or Corollary 3.6.

Also, from [4: Example 6.6], one can see that ks-spaces or *wcc*-spaces are not necessarily preserved by two-to-one open maps.

Theorem 3.11. *Let $f : X \longrightarrow Y$ be a quasi-perfect map. If Y is a wcc-space, then X is a wcc-space.*

Corollary 3.12. *Let $f : X \longrightarrow Y$ be a quasi-perfect map. If X is a γ, T_2-space and Y is a wcc-space, then both X and Y are metrizable.*

Corollary 3.13 [10]. *Let$f : X \longrightarrow Y$ be a quasi-perfect map. If Y is a β-space, then X is β-space.*

Note that [32] the closed image of a β, T_1-space is β, which is also proved in Case 1 of Theorem 2.1.

Corollary 3.14. *Let $f : X \longrightarrow Y$ be a quasi-perfect map. If Y is a q-space, then X is a q-space.*

Although the perfect pre-imsge of a $Nagata$ space is not necessarily $Nagata$ by Example 2.3, the following theorem follows from Theorems 3.5, 3.11 and Corollary 3.14.

Theorem 3.15. *Let $f : X \longrightarrow Y$ be a quasi-perfect map. If Y is a wN-space, then X is a wN-space.*

The first statement of the following corollary makes slightly better [33: Theorem 12; 32: Theorem 3]

Corollary 3.16. *Let f be a closed map from a T_1-space X onto a q-space Y. Then the following statements hold.*
(1) *If X is a wcc-space, then Y is wN.*
(2) *If X is a contraconvergent space, then Y is Nagata.*

Let f be a closed map from a space X onto a q-space Y. Then, it is well-known that $\partial f^{-1}(y)$ is countably compact if X is normal [26] or countably paracompact [32]. Although *wcc*-spaces are not necessarily countably paracompact, the similar result follows.

Theorem 3.17. *Let $f : X \longrightarrow Y$ be a closed map. If X is a wcc, T_1-space and Y is a q-space, then $\partial f^{-1}(y)$ is countably compact for any $y \in Y$.*
Moreover if X is semistratifiable T_2, then $\partial f^{-1}(y)$ is compact.

Lutzer [22: Example 4.3] showed that there exists the perfect map from the $Nagata$ space which is not of countable type to the space which is not q. In Theorem 3.17 we consider the conditions for Y to be a q-space when $\partial f^{-1}(y)$ is compact. For that, we present the following two properties of a space X.

(a) X is a *wcc*, semistratifiable T_1-space of countable type, where X is of *countable type* if every compact subset of X has a countable character.

(b) X is a $Nagata$ space of countable type.

Condition (a) is strictly weaker than (b) since the space X in the below Example 4.9 satisfies (a) but it is not ks.

Corollary 3.18. *Suppose that X satisfies (a) or (b). Then for a closed map $f : X \longrightarrow Y$, the following conditions are equivalent.*

(1) *Y satisfies (a) or (b), respectively.*
(2) *Y is a q-space.*
(3) *$\partial f^{-1}(y)$ is compact for any $y \in Y$.*

4. Strongly α-structures

Definition 4.1. A space X is called *strongly* α if for any $x \in X$ and each $n \in \mathbf{N}$, there exists an open neighbourhood $g_n(x)$ of x such that

(a) $\cap_{n\geq 1}\overline{g_n(x)} = \{x\}$ and,
(b) $g_n(y) \subset g_n(x)$ if $y \in g_n(x)$.

Here, we can assume that the sequence $\{g_n(x)\}$ is decreasing.

Evidently, every strongly α-space is a T_2-space.

Let consider the following properties $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$ of a space.

$(\mathcal{P}_1)$ There exists a sequence $\{\mathcal{F}_n\}$ consisting of closure-preserving closed covers of a space X such that, if $x \neq y$ then there exists $m \in \mathbf{N}$ such that for any $p \in X$, $x \notin F_p$ or $y \notin F_p$ for some $F_p \in \mathcal{F}_m$ with $p \in F_p$.

$(\mathcal{P}_2)$ There exists a sequence $\{\mathcal{U}_n\}$ consisting of point-finite open covers of a space X such that, if $x \neq y$ then there exists $m \in \mathbf{N}$ such that for any $p \in X$, $x \in V$ or $y \in V$ for some $V \in \mathcal{U}_m$ with $p \notin V$.

Definition 4.2. A space X is said to have a *strong G_δ-diagonal* if X has a sequence $\{\mathcal{G}_n\}$ of open covers such that whenever $x \neq y$, there exists $m \in \mathbf{N}$ satisfying that $x \notin st(p, \mathcal{G}_m)$ or $y \notin st(p, \mathcal{G}_m)$ for any $p \in X$. The sequence $\{\mathcal{G}_n\}$ is called a *strong G_δ-diagonal sequence.*

Definition 4.3. A space X is called *subparacompact*(*metacompact*) if every open cover of X has a σ-discrete closed refinement (a point finite open refinement).

Every semistratifiable space is subparacompact [5; 12] and every subparacompact space with G_δ-diagonal is α [16].

In the realm of paracompact T_2-spaces, the existence of a strong G_δ-diagonal is equivalent to it of a G_δ-diagonal.

Proposition 4.4. (1) *If X satisfies $(\mathcal{P}_1)$, then X is strongly α.*

(2) *If X satisfies $(\mathcal{P}_2)$, then X is strongly α.*

(3) *If X is a subparacompact space with a strong G_δ-diagonal, then X satisfies $(\mathcal{P}_1)$.*

(4) *If X is a metacompact space with a strong G_δ-diagonal, then X satisfies $(\mathcal{P}_2)$.*

(5) *A submetrizable space X satisfies $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$ and hence, every paracompact T_2-space with a G_δ-diagonl also satisfies $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$.*

Theorem 4.5. *Every developable T_2-space X is a strongly α-space with a strong G_δ-diagonal.*

Note that every stratifiable space or Sorgenfrey line is strongly α, because it is a paracompact T_2-space with a G_δ-diagonal.

Theorm 4.6. *Let $f : X \longrightarrow Y$ be a finite-to-one open closed map. If X is a strongly α-space, then Y is strongly α.*

Proposition 4.7. *For a strongly α-space X, the following statements hold.*
(1) *If X is a wcc-space, then X is k-semistratifiable.*
(2) *If X is a $w\gamma$-space, then X is γ.*
(3) *If X is a $w\theta$-space, then X is θ.*

Theorem 4.8. *For a space X, the following conditions are equivalent.*
(1) *X is a metrizable space.*
(2) *X is a strongly α, wM-space.*
(3) *X is a strongly α, wcc, $w\theta$-space.*
(4) *X is a strongly α, ks, $w\theta$-space.*

Example 4.9. There exists a compact γ, α (hence, semistratifiable), T_1-space X which is neither ks nor strongly α. Moreover, X is of countable type.

Let X be a space $X = (\mathbf{N}, \mathcal{O})$ with the topology $\mathcal{O} = \{G \subset \mathbf{N} \,||\mathbf{N} \setminus G| < \omega_0\}$. Then X is a compact T_1-space which is not T_2. To see that X is α, let $A_n = \{k \mid k \geq n\}$ for each $n \in \mathbf{N}$. For each $x \in X$ and each $n \in \mathbf{N}$, let $g_n(x) = \{x\} \cup A_n$. Then $\mathcal{G} = (\{g_n(x)\} \mid x \in X)$ is an α-structure. Therefore X is semistratifiable [16: Theorem 5.2]. We show that $\mathcal{G}$ is a γ-structure. Let $y_n \in g_n(p)$, $x_n \in g_n(y_n)$ for each $n \in \mathbf{N}$. Since $\mathcal{G}$ is an α-structure, $x_n \in g_n(p)(n \geq 1)$. If $p \notin \{x_n \mid n \geq m\}$ for some $m \in \mathbf{N}$, then $x_n \in A_n(n \geq m)$ and $\{x_n\}_{n \geq m}$ is infinite. Hence $\{x_n\}_n$ converges to p. Next, if X is ks, then X is $Nagata$ [34: Theorem 2] and if X is strongly α, then X is T_2, which are contradictions. Last, we prove that X is of countable type. For any compact subset K of X, let $\mathcal{A} = \{X \setminus F \mid F$ is any finite subset of $X \setminus K\}$. Then $\mathcal{A}$ is a countable base of K.

5. γ-SPACES

The irreducible closed images of γ-spaces are not necesarily γ (even q) [26: Example 10.1] (or [6: Problem 5.5.12]). However, Gittings [9: Theorem 4.1] showed that a finite-to-one open image of a γ-space (a $w\gamma$-space) also is a γ-space (a $w\gamma$-space, respectively). In this section, we study open closed images or quasi-perfect images of γ-spaces or $w\gamma$-spaces. For that reason, we need the following lemma.

Lemma 5.1. *Let $\mathcal{G} = (\{g_n(x)\} \mid x \in X)$ be a g-structure of a space X. Then the following statements hold.*

(1) *$\mathcal{G}$ is a γ-structure if and only if the sequence $\{y_n\}$ has a cluster point x whenever $y_n \in g_n(x_n)(n \geq 1)$ and the sequence $\{x_n\}$ has a cluster point x [21] if and only if the sequence $\{y_n\} \longrightarrow x$ whenever $y_n \in g_n(x_n)(n \geq 1)$ and the sequence $\{x_n\} \longrightarrow x$ [34].*

(2) *$\mathcal{G}$ is a $w\gamma$-strucyure if and only if the sequence $\{y_n\}$ has a cluster point whenever $y_n \in g_n(x_n)(n \geq 1)$ and the sequence $\{x_n\}$ has a cluster point.*

Theorem 5.2. *Let $f : X \longrightarrow Y$ be a quasi-perfect map. Then the following statements hold.*

(1) *If X is a γ-space, then Y is γ.*
(2) *If X is a $w\gamma$-space, then Y is $w\gamma$.*

Example 5.3. θ-spaces are not necessarily preserved by quasi-perfect maps.

Although Example 2.3 asserts that the perfect pre-image of a γ-space is not necessarily γ, the quasi-perfect pre-images of $w\gamma$-spaces also are $w\gamma$.

Theorem 5.4. *Let $f : X \longrightarrow Y$ be a quasi-perfect map. If Y is a $w\gamma$-space, then X is a $w\gamma$-space.*

Last, we consider the open closed images of γ-spaces or $w\gamma$-spaces.

Theorem 5.5. *Let $f : X \longrightarrow Y$ be an open closed map. Then the following statements hold.*
(1) *If X is a γ, T_1-space, then Y is γ.*
(2) *If X is a $w\gamma$, T_1-space, then Y is $w\gamma$.*

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Department of Mathematics, Okayama University, Tsushima, Okayama, Japan
E-mail address: yoshiopka@math.okayama-u.ac.jp

参加者リスト（五十音順）

新井　達也	arai@math.tsukuba.ac.jp	筑波技術短期大学
池田 義人	yoshito@u-gakugei.ac.jp	東京学芸大学
伊藤　宗彦	ito@is.kochi-u.ac.jp	高知大学
今井　淳	aimai@math.kyushu-u.ac.jp	京都大学（研究生）
大田　春外	echohta@ipc.shizuoka.ac.jp	静岡大学
奥山　晃弘	okuyama@kobe-u.ac.jp	神戸女子短期大学
小野　仁	tsjono@eng.shizuoka.ac.jp	静岡大学
加藤 久男	hisakato@sakura.cc.tsukuba.ac.jp	筑波大学
川口　慎二		筑波大学（院生）
木村　孝	kimura@post.saitama-u.ac.jp	埼玉大学
家本　宣幸	nkemoto@cc.oita-u.ac.jp	大分大学
小竹　義朗	kotake@edu.gunma-u.ac.jp	群馬大学
菰田智恵子	komodac@fne.freeserve.ne.jp	静岡大学（院生）
小山　晃	koyama@cc.osaka-kyoiku.ac.jp	大阪教育大学
酒井 克郎	sakaiktr@sakura.cc.tsukuba.ac.jp	筑波大学
酒井　政美	msakai@cc.kanagawa-u.ac.jp	神奈川大学
嶋根　紀仁	shimane@kagoshima-ct.ac.jp	鹿児島高専
清水　一生	issei@kurt.scitec.kobe-u.ac.jp	神戸大学（院生）
Dmitri Shakhmatov	dmitri@dpc.ehime-u.ac.jp	愛媛大学
鈴木　晃	sakira@kobe-u.ac.jp	神戸大学
高橋　真	makoto@kobe-u.ac.jp	神戸大学
田中　秀典	htanaka@cc.osaka-kyoiku.ac.jp	大阪教育大学
知念 直紹	naochin@math.tsukuba.ac.jp	筑波大学
千葉　慶子	smktiba@ipc.shizuoka.ac.jp	静岡大学
寺田　敏司	terada@math.sci.ynu.ac.jp	横浜国立大学
Artur H. Tomita	tomita@ime.usp.br	Sao Paulo University
友安　一夫	tomoyasu@cc.miyakonojo-nct.ac.jp	都城高専
新田　眞一	nitta@sg-jc.ac.jp	四条畷学園短期大学
平田　康史	yhira@jb3.so.ne.jp	筑波大学（院生）
深石 博夫	fukaishi@ed.kagawa-u.ac.jp	香川大学
細渕　昌美	mhsbc@kasei-gakuin.ac.jp	東京家政学院大学
渕野 昌	fuchino@isc.chubu.ac.jp	中部大学
Joerg Brendle	brendle@kurt.scitec.kobe-u.ac.jp	神戸大学
保科　隆雄	takaohsn@math.tsukuba.ac.jp	筑波大学
堀内　清光	horiuchi@konan-u.ac.jp	甲南大学
牧　春夫	makih@pop12.odn.ne.jp	佐賀大学
三村　護	mimura@math.okayama-u.ac.jp	岡山大学
宮嵜 和美	BZQ22206@nifty.ne.jp	大阪電通大学
宮田 任寿	tmiyata@kobe-u.ac.jp	神戸大学
三輪 拓夫	miwa@math.shimane-u.ac.jp	島根大学
牟 磊（Mou Lei）		静岡大学（研究生）
森下　和彦	mrst@ashitech.ac.jp	足利工業大学
矢ヶ崎　達彦	yagasaki@ipc.kit.ac.jp	京都工繊大学
矢口　雅人		筑波大学（院生）
矢島　幸信	yuki@cc.kanagawa-u.ac.jp	神奈川大学
矢田部　俊介	yatabe@kobe.email.ne.jp	神戸大学（院生）

参加者リスト（五十音順）（つづき）

山内　貴光		筑波大学（院生）
山崎　薫里	kaori@math.tsukuba.ac.jp	筑波大学
山田　耕三	eckyama@ipc.shizuoka.ac.jp	静岡大学
吉岡　巌	yoshioka@math.okayama-u.ac.jp	岡山大学
渡辺　正	tadashi@yamaguchi-u.ac.jp	山口大学

Participants

Tatsuya Arai	arai@math.tsukuba.ac.jp	Tsukuba Junior College of Tech
Keiko Chiba	smktiba@ipc.shizuoka.ac.jp	Shizuoka Univ
Naotsugu Chinen	naochin@math.tsukuba.ac.jp	Tsukuba Univ
Sakaé Fuchino	fuchino@isc.chubu.ac.jp	Chunu Univ
Hiroo Fukaishi	fukaishi@ed.kagawa-u.ac.jp	Kagawa Univ
Yasushi Hirata	yhira@jb3.so.ne.jp	Tsukuba Univ (grad student)
Kiyomitsu Horiuchi	horiuchi@konan-u.ac.jp	Konan Univ
Takao Hoshina	takaohsn@math.tsukuba.ac.jp	Tsukuba Univ
Masami Hosobuchi	mhsbc@kasei-gakuin.ac.jp	Tokyo Kasei Gakuin Univ
Yoshito Ikeda	yoshito@u-gakugei.ac.jp	Tokyo Gakugei Univ
Atsushi Imai	aimai@math.kyushu-u.ac.jp	Kyoto Univ（Research Scholar）
Munehiko Ito	ito@is.kochi-u.ac.jp	Kochi Univ
Brendle Joerg	brendle@kurt.scitec.kobe-u.ac.jp	Kobe Univ
Hisao Kato	hisakato@sakura.cc.tsukuba.ac.jp	Tsukuba Univ
Shinji Kawaguchi		Tsukuba Univ (grad student)
Nobuyuki Kemoto	nkemoto@cc.oita-u.ac.jp	Oita Univ
Takashi Kimura	kimura@post.saitama-u.ac.jp	Saitama Univ
Chieko Komoda	komodac@fne.freeserve.ne.jp	Shizuoka Univ (grad student)
Yoshiro Kotake	kotake@edu.gunma-u.ac.jp	Gumma Univ
Akira Koyama	koyama@cc.osaka-kyoiku.ac.jp	Osaka Kyoiku Univ
Mou Lei		Shizuoka Univ (Research Scholar)
Haruo Maki	makih@pop12.odn.ne.jp	Saga Univ
Mamoru Mimura	mimura@math.okayama-u.ac.jp	Okayama Univ
Takahisa Miyata	tmiyata@kobe-u.ac.jp	Kobe Univ
Kazumi Miyazaki	BZQ22206@nifty.ne.jp	Osaka Dentsu Univ
Takuo Miwa	miwa@math.shimane-u.ac.jp	Shimane Univ
Kazuhiko Morishita	mrst@ashitech.ac.jp	Ashikaga Inst of Tech
Shinichi Nitta	nitta@sg-jc.ac.jp	Shijonawate-Gakuen Junior College
Haruto Ohta	echohta@ipc.shizuoka.ac.jp	Shizuoka Univ
Teruhiko Okuyama	okuyama@kobe-u.ac.jp	Kobe Women Junior College
Jin Ono	tsjono@eng.shizuoka.ac.jp	Shizuoka Univ
Katsuro Sakai	sakaiktr@sakura.cc.tsukuba.ac.jp	Tsukuba Univ
Masami Sakai	msakai@cc.kanagawa-u.ac.jp	Kanagawa Univ
Norihito Shimane	shimane@kagoshima-ct.ac.jp	Kagoshima College of Tech
Issei Shimizu	issei@kurt.scitec.kobe-u.ac.jp	Kobe Univ (grad student)
Dmitri Shakhmatov	dmitri@dpc.ehime-u.ac.jp	Ehime Univ
Akira Suzuki	sakira@kobe-u.ac.jp	Kobe Univ
Makoto Takahashi	makoto@kobe-u.ac.jp	Kobe Univ
Hidenori Tanaka	htanaka@cc.osaka-kyoiku.ac.jp	Osaka Kyoiku Univ
Toshiji Terada	terada@math.sci.ynu.ac.jp	Yokohama National Univ
Artur H. Tomita	tomita@ime.usp.br	Sao Paulo University
kazuo Tomoyasu	tomoyasu@cc.miyakonojo-nct.ac.jp	Miyakonojo College of Tech
Tatsuhiko Yagasaki	yagasaki@ipc.kit.ac.jp	Kyoto Inst of Tech
Masato Yaguchi		Tsukuba Univ (grad student)
Yukinobu Yajima	yuki@cc.kanagawa-u.ac.jp	Kanagawa Univ
Shunsuke Yatabe	yatabe@kobe.email.ne.jp	Kobe Univ (grad student)

Participants (continue)

Takamitsu Yamauchi		Tsukuba Univ (grad student)
Kaori Yamazaki	kaori@math.tsukuba.ac.jp	Tsukuba Univ
Kohzo Yamada	eckyama@ipc.shizuoka.ac.jp	Shizuoka Univ
Iwao Yoshioka	yoshioka@math.okayama-u.ac.jp	Okayam Univ
Tadashi Watanabe	tadashi@yamaguchi-u.ac.jp	Yamaguchi Univ