Hyperspaces with the Hausdorff Metric

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In this talk, we represent the rersult obtained in [2].

Let $X = (X, d)$ be a metric space. The set of all non-empty closed sets in X is denoted by $\mathrm{Cld}(X)$. On the subset $\mathrm{Bdd}(X) \subset \mathrm{Cld}(X)$ consisting of bounded closed sets in X, we can define the *Hausdorff metric* d_H as follows:

$$d_H(A, B) = \max\left\{\sup_{x\in B} d(x, A),\ \sup_{x\in A} d(x, B)\right\},$$

where $d(x, A) = \inf_{a\in A} d(x, a)$. We denote the metric space $(\mathrm{Bdd}(X), d_H)$ by $\mathrm{Bdd}_H(X)$. On the whole set $\mathrm{Cld}(X)$, we allow $d_H(A, B) = \infty$, but d_H induces the topology of $\mathrm{Cld}(X)$ like a metric does. The space $\mathrm{Cld}(X)$ with this topology is denoted by $\mathrm{Cld}_H(X)$. When X is bounded, $\mathrm{Cld}_H(X) = \mathrm{Bdd}_H(X)$. Even though X is unbounded, $\mathrm{Cld}_H(X)$ is metrizable. Indeed, let $\bar{d}$ be the metric on X defined by $\bar{d}(x, y) = \min\{1, d(x, y)\}$. Then, $\bar{d}_H$ is an admissible metric of $\mathrm{Cld}_H(X)$. It should be noted that each component of $\mathrm{Cld}_H(X)$ is contained in $\mathrm{Bdd}(X)$ or in the complement $\mathrm{Cld}(X) \setminus \mathrm{Bdd}(X)$. Thus, $\mathrm{Bdd}_H(X)$ is a union of components of $\mathrm{Cld}_H(X)$. On each component of $\mathrm{Cld}_H(X)$, d_H is a metric even if it is contained in $\mathrm{Cld}(X) \setminus \mathrm{Bdd}(X)$. Then, we regard every component of $\mathrm{Cld}_H(X)$ as a metric space with d_H.

In case X is compact, it is well-known as Wojdysławski's theorem [4] that $\mathrm{Cld}_H(X)$ $(= \mathrm{Bdd}_H(X))$ is an ANR (an AR), if and only if X is locally connected (connected and locally connected) where an ANR (an AR) means an absolute neighborhood retract (an absolute retract) for metrizable spaces. However, this theorem does not hold if X is non-compact. Here, we construct a metric AR X such that $\mathrm{Cld}_H(X)$ is not an ANR and give a condition on X such that $\mathrm{Cld}_H(X)$ is an ANR (actually, each component of $\mathrm{Cld}_H(X)$ is a uniform AR in the sense of E. Michael [3]). Due to our result, $\mathrm{Cld}_H(X)$ can be an ANR even if X is not locally connected. Our condition on X such that $\mathrm{Cld}_H(X)$ is an ANR weakens the one of C. Costantini and W. Kubiś [1].

References

[1] C. Costantini and W. Kubiś, *Paths in hyperspaces*, preprint.

[2] M. Kurihara, K. Sakai and M. Yaguchi, *Hyperspaces with the Hausdorff metric and uniform ANR's,* preprint.

[3] E. Michael, *Uniform AR's and ANR's,* Compositio Math. **39** (1979), 129–139.

[4] M. Wojdysławski *Rétractes absoulus et hyperespaces des continus,* Fund. Math., **32** (1939), 184–192.