Hereditarily separable group topologies on Abelian groups

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All group topologies in this talk are considered to be Hausdorff (and thus Tychonoff). Recall that a topological space X is:

Lindelöf if every open cover of X has a countable subcover,

(countably) compact if every (countable) open cover of X has a finite subcover,

pseudocompact if every real-valued continuous function defined on X is bounded,

hereditarily separable if every subspace of X is separable (in the subspace topology).

An *S-space* is a hereditarily separable regular space that is not Lindelöf.

Recall that a topological group G is *precompact*, or *totally bounded*, if G is (topologically and algebraically isomorphic to) a subgroup of some compact group. We have a chain

compact $\rightarrow$ countably compact $\rightarrow$ pseudocompact $\rightarrow$ precompact

of compactness-like conditions for topological groups.

Let $\mathbf{c}$ denote the cardinality of the continuum. Using forcing we produce a model of $ZFC + CH$ with $2^{\mathbf{c}}$ "arbitrarily large" and, in this model, obtain a characterization of Abelian groups G (necessarily of size at most $2^{\mathbf{c}}$) which admit:

(i) a hereditarily separable group topology,

(ii) a group topology making G into an S-space,

(iii) a hereditarily separable group topology that is either precompact, or pseudocompact, or countably compact (and which can be made to contain no non-trivial convergent sequences),

(iv) a group topology making G into an S-space that is either precompact, or pseudocompact, or countably compact (and which also can be made without non-trivial convergent sequences, if necessary).

As a by-product, we completely describe the algebraic structure of Abelian groups of size at most $2^{\mathbf{c}}$ which possess, at least consistently, a countably compact group topology (without non-trivial convergent sequences, if desired).

We also put to rest a 1980 problem of van Douwen about the cofinality of size of countably compact Abelian groups.