Extension, Insertion and Selection

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We give a variety of insertion theorems by introducing lower and upper semi-continuity of a map to the Banach space $C_0(Y)$ for a space Y. As usual, the symbol $c_0(\lambda)$ is used to denote $C_0(Y)$ in case Y is a discrete space with cardinality λ.

Theorem 1. *For an infinite cardinal, the following conditions on a space X are equivalent*:

(1) *Every point-finite open cover $\mathcal{U}$ of X, with $|\mathcal{U}| \leq \lambda$, is normal.*
(2) *For every two maps $g, h : X \to c_0(\lambda)$ such that g is upper semi-continuous, h is lower semi-continuous and $g \leq h$, there exists a continuous map $f : X \to c_0(\lambda)$ such that $g \leq f \leq h$.*

Theorem 2. *The following conditions on a space X are equivalent*:

(1) *X is paracompact.*
(2) *For every space Y and every upper semi-continuous map $g : X \to C_0(Y)$, there exists a continuous map $f : X \to C_0(Y)$ such that $g \leq f$.*

In Theorem 1 (Theorem 2) the implication (1) $\Rightarrow$ (2) follows from Kandô-Nedev's (Michael's) selection theorem. The following problem remains open: Is a space X paracompact provided for every space Y and every two maps $g, h : X \to C_0(Y)$ such that g is upper semi-continuous, h is lower semi-continuous and $g \leq h$, there exists a continuous map $f : X \to C_0(Y)$ with $g \leq f \leq h$?

Sandwich-like extension theorem. *Let λ be an infinite cardinal and A a subspace of a space X. Then, A is P^{λ}-embedded in X if and only if for every two maps $g, h : A \to c_0(\lambda)$ such that g is upper semi-continuous, h is lower semi-continuous and $g \leq h$, there exists a continuous map $f : X \to c_0(\lambda)$ such that $g \leq f|_A \leq h$.*

$$\begin{array}{ccccc} & & \text{Selection} & & \\ & & \downarrow & & \\ \text{Insertion} & \longleftarrow & \text{Sandwich-like extension} & \longrightarrow & \text{Extension} \end{array}$$

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