On the property "SEP" of partial orderings

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Abstract

A partial ordering $(P, \leq)$ is said to have *the weak Freese-Nation property* if there is a mapping $f : P \to [P]^{\leq\aleph_0}$ such that:

(*) For any $p,\ q \in P$ with $p \leq q$ there is $r \in f(p) \cap f(q)$ such that $p \leq r \leq q$.

A mapping f as above is called a *weak Freese-Nation mapping* on P.

The weak Freese-Nation property is a weakening of the following property: a partial ordering $(P, \leq)$ is said to have *the Freese-Nation property* if there is a mapping $f : P \to [P]^{<\aleph_0}$ with (*) as above.

This property has its topological counter-part. By a theorem of Lutz Heindorf, a Boolean algebra has the Freese-Nation property if and only if its topological dual is κ-metrizable, a notion defined by Schepin in [7] (see [5]). While κ-metrizability is such a strong notion that a Boolean algebra with this property is quite similar to a projective algebra, the weak Freese-Nation property is much weaker so that, for example, even the powerset algebra $\mathcal{P}(\omega)$ can have this property [3]. In [1], it is shown that a model of set-theory where $\mathcal{P}(\omega)$ has the weak Freese-Nation property behaves very similar to a Cohen model.

Several further weakenings of weak Freese-Nation property have been studied. One of them is so-called "SEP" introduced by Juhász and Kunen [6]. For a Boolean algebra, this property can be formulated as follows:

For Boolean algebras A and B such that $A \leq B$, $A \leq_{sep} B$ if and only if, for every $b \in B$ and every uncountable set $T \subseteq A\restriction b$, there is $a \in A\restriction b$ such that $\{c \in T : c \leq a\}$ is uncountable.

For a cardinal χ let $\mathcal{M}_\chi$ be the set of elementary submodels M of $\mathcal{H}(\chi)$ such that $|\,M\,| = \aleph_1$ and $[M]^{\aleph_0} \cap M$ is cofinal in $[M]^{\aleph_0}$.

A Boolean algebra B has the *SEP* if $\{M \in \mathcal{M}_\chi : B \cap M \leq_{sep} B\}$ is cofinal in $[\mathcal{H}(\chi)]^{\aleph_0}$ (with respect to $\subseteq$) for all sufficiently large χ.

We show that this notion is actually a weakening of the weak Freese-Nation property. A characterization given to show this naturally generalizes to a property for an arbitrary partial ordering.

In ZFC, we can construct a Boolean algebra which has the SEP but not the weak Freese-Nation property. Likewise, there is a model of set-theory where $\mathcal{P}(\omega)$ has the SEP but not the weak Freese-Nation property. Nevertheless, it can be shown that,

under the SEP of $\mathcal{P}(\omega)$, many consequences of the weak Freese-Nation property of $\mathcal{P}(\omega)$ still hold. In particular, most of common cardinal invariants from the set-theory of the reals under the SEP of $\mathcal{P}(\omega)$ takes the same value as in a Cohen model with the same value of $2^{\aleph_0}$.

It is tempting to define the following variant of SEP:

$$\{M \in \mathcal{M}_\chi : B \cap M \leq_{sep} B\} \text{ is stationary in } [\mathcal{H}(\chi)]^{\aleph_0} \text{ for all sufficiently large } \chi.$$

However, we show that this property is equivalent to the original SEP under an appropriate interpretation of "sufficiently large".

The results presented in this talk are obtained in a joint research with Stefan Geschke [2].

References

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