The Baire space ordered by eventual domination: results and problems

Jörg Brendle

December 18, 2002

The *Baire space* ω^ω is the set of all functions from the natural numbers ω to ω, equipped with the product topology of the discrete topology. Given $f, g \in \omega^\omega$ say that *g eventually dominates f* ($f \leq^* g$ in symbols) if $f(n) \leq g(n)$ holds for all but finitely many $n \in \omega$. We shall study several kind of spectra for the preorder $(\omega^\omega, \leq^*)$.

For a given preorder $(P, \leq)$ (that is, $\leq$ is reflexive and transitive, but not necessarily antisymmetric), Fuchino and Soukup defined the following four spectra.

(i) the *unbounded set spectrum* $\mathfrak{S}^\uparrow(P)$, the set of all cardinals κ such that there is an unbounded increasing chain of length κ in P;

(ii) the *hereditarily unbounded set spectrum* $\mathfrak{S}^h(P)$, the set of all cardinals κ such that there is $A \subseteq P$ of size κ such that all subsets of A of size κ are unbounded in P while all subsets of A of size less than κ are bounded in P;

(iii) the *unbounded set spectrum* $\mathfrak{S}(P)$, the set of all cardinals κ such that there is unbounded $A \subseteq P$ of size κ such that all subsets of A of size less than κ are bounded in P;

(iv) the *unbounded family spectrum* $\mathfrak{S}^s(P)$, the set of all cardinals κ such that there is a family $\mathcal{F} \subseteq \mathcal{P}(P)$ of size κ with $\bigcup \mathcal{F}$ being unbounded while for all $\mathcal{G} \subseteq \mathcal{F}$ of size less than κ, $\bigcup \mathcal{G}$ is bounded.

Clearly $\mathfrak{S}^\uparrow(P) \subseteq \mathfrak{S}^h(P) \subseteq \mathfrak{S}(P) \subseteq \mathfrak{S}^s(P)$.

We shall study the connection between these spectra for $(P, \leq) = (\omega^\omega, \leq^*)$. It is easy to see that $\mathfrak{b} = \min \mathfrak{S}^\uparrow(\omega^\omega, \leq^*) = \min \mathfrak{S}^s(\omega^\omega, \leq^*)$. In joint work with Laberge (*Forcing tightness in products of fans*, Fund. Math. 150 (1996)) we obtained a model where $\aleph_2 \in \mathfrak{S}^h(\omega^\omega, \leq^*) \setminus \mathfrak{S}^\uparrow(\omega^\omega, \leq^*)$ and $\mathfrak{c} = \aleph_2$.

Recently, we also proved that $\aleph_2 \in \mathfrak{S}(\omega^\omega, \leq^*) \setminus \mathfrak{S}^h(\omega^\omega, \leq^*)$ is consistent. In fact, we have two models for this, an easier one in which $\mathfrak{c} = \aleph_3$, and a more sophisticated one with $\mathfrak{c} = \aleph_2$.

The purpose of this talk is to present the main ideas of these three results. We conjecture that similar techniques can be used to get a model where $\aleph_2 \in \mathfrak{S}^s(\omega^\omega, \leq^*) \setminus \mathfrak{S}(\omega^\omega, \leq^*)$, but so far we have been unable to prove this.